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NOTES ON MAGNETO-HYDRODYNAMICS

V

THEORY OF MAXWELL'S EQUATIONS WITHOUT DISPLACEMENT CURRENT

by

A. A. Blank
K. O. Friedrichs
H. Grad

November 1, 1957

Institute of Mathematical Sciences

NEW YORK UNIVERSITY

NEW YORK, NEW YORK

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Contract No. AT(30-1)-1480

PREFACE

We use the term pre-Maxwell equations to signify a formalization of the electromagnetic theory which existed prior to Maxwell's introduction of the concept of displacement current. The purpose of this report is to give a general descriptive survey of the theory of these equations which is aimed primarily at applications. The pre-Maxwell equations yield satisfactory approximations to solutions of electromagnetic problems in a number of fields; chiefly, in classical network and electrical machinery problems, and in large parts of fluid magnetics. The elimination of electromagnetic wave propagation at the very outset in problems where it does not play an important role frequently results in great simplification. Moreover, in formulations of fluid magnetics which use non-relativistic fluid dynamic equations, the pre-Maxwell equations, being Galilean invariant, are in some respects even more appropriate than Maxwell's equations.

The system of pre-Maxwell equations has its own intrinsic mathematical interest because of properties which are entirely different from those of Maxwell's equations. It is possible to give a rather complete existence theory for the system of pre-Maxwell equations

by using the properties of harmonic vectors. Many of the results quoted here have been proved in recent years[†], however we have not hesitated to label as "theorem" a number of additional results which, although not yet rigorously proved, seem susceptible of proof by the same techniques.

The valuable assistance of Hanan Rubin in preparing an early draft of this report is gratefully acknowledged. The fundamentals of this theory were originally outlined in 1954.

H. G.

[†] For references see, e.g., K. O. Friedrichs, "Differential forms on Riemannian manifolds," Comm. on Pure and Appl. Math., VIII, 551-590 (1955).

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NOTES ON MAGNETO-HYDRODYNAMICS - NUMBER V

Theory of Maxwell's Equations without Displacement Current

A. A. Blank, K. O. Friedrichs, H. Grad

1. Introduction

In many applications of the electromagnetic equations, e.g. to fluid magnetics, the displacement current term in Maxwell's equations is considered negligibly small and therefore is omitted. The truncated electromagnetic system[1],

$$(1.1) \quad \begin{cases} \frac{\partial B}{\partial t} + \operatorname{curl} E = 0 & \operatorname{div} E = \frac{q}{\kappa} \\ \operatorname{curl} B = \mu J & \operatorname{div} B = 0 \end{cases},$$

obtained in this way will be called the pre-Maxwell system because it represents the content of electromagnetic theory before the introduction of displacement current by Maxwell.

These equations are an appropriate formulation of electromagnetics in the conventional engineering analysis of rotating machinery. In fluid magnetics, when the fluid equations are given in non-relativistic form, the pre-Maxwell system would seem the more appropriate for its invariance under Galilean transformation[2], compared with the Lorentz invariance of Maxwell's equations. This usage will not only prove less awkward but may possibly be more accurate.

---

[1] In rationalized units.

[2] Galilean invariance is obtained by transforming as vectors the quantities $E_u = E + u \times B$ (polar) and B (axial). E_u and B are to be interpreted as the electric and magnetic field intensities as observed in the given frame moving with velocity u.

It is usually assumed that the equations (1.1) represent an approximate description of a certain class of solutions of the complete system of Maxwell's equations. Actually, although the difference in the two systems of equations may appear slight, it cannot be assumed without analytical investigation that a solution of one will approximate a solution of the other. In fact, despite the similarity of the two systems, and even though it is true that the solutions of one may under certain circumstances approximate solutions of the other, the mathematical properties of the two systems differ radically. The difference between the two systems is closely comparable to the difference between Laplace's equation and the wave equation. For example, the pre-Maxwell system admits essentially only boundary values, where Maxwell's system admits not only boundary values, but initial values also.

To gloss over the differences between the two systems may be serious. For example, the system (1.1) is sometimes used together with the equation of conservation of charge $\partial q/\partial t = \text{div } J$, despite the fact that this relation will usually contradict the equation $\text{div } J = 0$, a plain consequence of the equations (1.1). The pre-Maxwell system together with charge conservation is overdetermined and will in general have no solution at all. This, of course, does not preclude the possibility that a given solution of the pre-Maxwell system may be a very good approximation to some solution of Maxwell's equations which does satisfy conservation of charge. For a solution of (1.1) which is a satisfactory approximation to a solution of Maxwell's equations, it is to be expected that a slight additional current would suffice to give conservation of charge. Nonetheless, the important mathematical point is that conservation of charge must not be imposed upon the system (1.1), for it will generally create an insoluble problem.

A crucial question which should be asked in connection with a given system of equations is whether a solution exists corresponding to every physically relevent problem. Equally important is the knowledge that there is no ambiguity, that the conditions of a problem determine its solution uniquely. If a stated problem admits precisely one solution it is said to be well-posed. A well-posed problem for the Maxwell system is of a character rather different from that of a well-posed problem for the pre-Maxwell system. For the system (1.1) a well-posed problem is, except for slight modifications, essentially a pure boundary value problem despite the appearance of the time derivative. We shall see that initial values in a pre-Maxwell problem may be assigned to certain discrete parameters and to E or B upon the boundary only, but not to E or B over the entire domain. The mathematical distinction between the systems is that the Maxwell system is of hyperbolic type while the pre-Maxwell system is almost elliptic in character. The difference is exemplified by the fact that no property akin to wave propagation may be attributed to the solutions of (1.1).

As the simplest illustration of these differences, we present the following typical well-posed problems for the two sets of equations. In each case we take $q = 0$, $J = 0$.

a). The domain is the interior of a sphere. For both systems of equations, the normal components, E_n and B_n, are specified arbitrarily as functions of time on the surface S of the sphere subject only to the conditions $\oint E_n dS = \oint B_n dS = 0$. In the Maxwell system, initial data E and B are given at $t = 0$ subject to the conditions $\operatorname{div} E = \operatorname{div} B = 0$. In the pre-Maxwell system no initial data can be imposed at all; the solution of (1.1) is already uniquely determined as a function of time by the boundary conditions.

b). The domain is again the interior of a sphere. In this case, however, the tangential component E_t of E, is specified on the surface of the sphere as a function of time. For Maxwell's equations, we would further specify E and B initially subject to $\operatorname{div} E = \operatorname{div} B = 0$. However, to make the pre-Maxwell problem well-posed, we may further specify only the initial values of B_n on the surface of the sphere (subject to $\oint B_n dS = 0$); nothing can be initially prescribed over the whole domain.

c). Let the domain be the interior of a torus. For the Maxwell equations the situation is exactly the same as in the preceding example for the sphere; we can prescribe the boundary condition $E_t = 0$ as well as initial values of E and B inside, restricted by $\operatorname{div} E = \operatorname{div} B = 0$. For the pre-Maxwell equations, in addition to the boundary condition $E_t = 0$ for all time, we can specify $B_n = 0$ initially, just as in the previous example, and also the initial value of the flux $\int B \cdot dS$ computed on a disc which cuts the torus.

While the Maxwell equations admit essentially arbitrary initial conditions, we see that the pre-Maxwell equations yield a unique solution under much less stringent initial conditions; in the above examples, these conditions involved the initial prescription of boundary values, or the value of a single parameter (the magnetic flux), and in one case there was no initial condition at all. The general situation is described in the theorems of Section 10.

The two sets of equations are further distinguished by the fact that the appropriate singular solutions of (1.1) have a character distinctly different from those of the Maxwell system. Corresponding to an electron at rest, there is a solution of either system,

$$E = \text{grad}\left(\frac{1}{r}\right) \quad , \qquad B = 0 \quad .$$

For Maxwell's equations there also exists a singular solution corresponding to a moving point charge. There is no corresponding solution of (1.1) in which the moving charge is considered to be a localized singular current source as well as a charge. This is clear, since (1.1) implies $\text{div } J = 0$. In other words, the physical picture of discrete moving charges must be smeared into an incompressible current flow before approaching the system

(1.1). A symptom of this state of affairs is the fact that the usual formula given for the magnetic force between two current elements (with no retardation) does not satisfy Newton's third law[3]; note that this is not a consequence of the neglect of radiation momentum. If the current elements are completed into closed loops (corresponding to the singular form of the condition div J = 0), the theory of (1.1), and consequently of this force formula, is applicable and yields equal action and reaction.

Turning again to the general theory of the pre-Maxwell system (1.1), we note that the form of the system suggests a certain simplification; namely, that a problem may be solved in two parts, first for B and then for E, or vice versa. For example, we may begin by obtaining B first from the system

$$(1.2) \qquad \begin{cases} \operatorname{curl} B = \mu J \\ \operatorname{div} B = 0 \end{cases} .$$

Then, utilizing the values of B obtained from (1.2), we may determine E from

$$(1.3) \qquad \begin{cases} \operatorname{curl} E = -\frac{\partial B}{\partial t} \\ \operatorname{div} E = \frac{q}{k} \end{cases} .$$

---

[3] E.g., Stratton, J. A., Electromagnetic Theory, McGraw-Hill, 1941, p. 266, Ex. 22.

The systems (1.2) and (1.3) are closely related to the potential equation. In fact, if $J = 0$, B has a scalar potential which satisfies the Laplace equation. The inhomogeneous equations (1.2) and (1.3) have a theory very similar to that of Laplace's equation. In particular, the natural problems are boundary value problems. In (1.2) if J is assumed given as a function of position and time, B may be fixed uniquely by boundary data and values of certain parameters called periods which later will receive their physical interpretation as currents or fluxes[4]. Once B is known, the vector E may then be found subject to the prescription of similar boundary conditions and periods. In this case the periods are naturally interpreted as electric charge or as e.m.f.[5].

Alternatively, it is possible to solve (1.1) by first solving for E in the system,

$$(1.4) \qquad \begin{cases} \operatorname{curl} \operatorname{curl} E = -\mu \dfrac{\partial J}{\partial t} \\ \operatorname{div} E = \dfrac{q}{K} \end{cases} ,$$

obtained by eliminating $\partial B/\partial t$ from the system (1.1). If E is determined uniquely as the solution of a well-posed problem for this system, then, without the prescription of additional boundary data, B may be found from

---

[4] A current is defined by the period $1/\mu \oint B \cdot dx$ defined on a closed curve; a flux by $\int B \cdot dS$ over a disc. These concepts will later be made more precise.

[5] The total charge within a surface is given by $K\oint E \cdot dS$. The e.m.f. or electromotive force along an arc is given by $\int E \cdot dx$.

$$(1.5) \qquad \frac{\partial B}{\partial t} = - \operatorname{curl} E \ ;$$

only certain initial properties of B will be needed. In this problem all the boundary data are imposed upon the vector E in contrast with the preceding problem where boundary conditions are divided between the two.

A well-posed problem for the system (1.1) will generally involve boundary and period data and initial values of such data. We have seen that the data appropriate to the method of solving sequentially (1.2) and (1.3) are not exactly the same as the data appropriate to the sequence (1.4), (1.5). Roughly, the difference lies in the way the subsidiary data have been apportioned to E and B. To formulate the general well-posed problem for (1.1) without reference to a particular method of solution, we shall characterize compatible sets of boundary, period, and initial data for both E and B.

In more complicated cases, as in fluid magnetics, J and q are not given a priori but are to be determined together with E and B after supplementing (1.1) by further equations. For example, Ohm's law connects J with E and B as well as the motion of the medium. Compatibility conditions between B and E may, of course, be modified in such cases. Frequently, it will turn out that the relation div $E = q/\kappa$ is not used in determining E, but only to obtain q after E has been found otherwise. This situation will be found to obtain for a perfectly conducting fluid, a case which will be discussed at length in subsequent notes.

The study of the pre-Maxwell system is reduced by (1.2) and (1.3) to the study of simpler systems having the form

$$(1.6) \qquad \begin{cases} \operatorname{div} Z = \sigma \\ \operatorname{curl} Z = \Xi \end{cases} \qquad\qquad (\operatorname{div} \Xi = 0).$$

As is often the case in linear analysis, many of the difficulties in the solution of the inhomogeneous system (1.6) may be assigned to the solution of the homogeneous system

$$(1.7) \qquad \begin{cases} \operatorname{div} Z = 0 \\ \operatorname{curl} Z = 0 \end{cases} .$$

The solutions of system (1.7) are the so-called harmonic vectors.

Harmonic fields may arise naturally in several ways. If $J = 0$ in some domain then B is harmonic there. If $q = 0$ and B is independent of time then E is harmonic. If the domain is simply connected, then a harmonic B is the gradient of a scalar potential and the problem of determining B reduces to that of solving the Laplace equation. This approach to the problem is quite general and any specific additional complications can only arise from the possible multi-valuedness of the potentials, essentially a topological difficulty. In applications to fluid magnetics where we are concerned with the magnetic field in a vacuum domain bounded by conductors, it is common to impose the normal boundary condition $B_n = 0$. When B is harmonic this condition implies that B must vanish identically in any simply-connected domain[6]. However, if the vacuum domain is bounded by a torus, this conclusion does not follow; there does exist a regular non-trivial field. For a precise analysis of these problems it is necessary to introduce a few topological concepts which clarify the pertinent intuitive notions.

In the first few sections, then, there will be a discussion of topological considerations in relation to the classical integral theorems of vector analysis. These results will next be used to formulate compatibility conditions in the statement of existence theorems for harmonic vectors and related fields. The theory of these fields will be pursued further to reach a variational formulation of the problems, an especially valuable approach for fluid magnetics. We present also a general existence theorem which goes somewhat beyond our immediate needs.

---

[6] The corresponding Neumann problem admits only constants as solutions, hence B is the gradient of a constant.

We present also, for its own intrinsic interest, a treatment of the two-dimensional case which is not merely a specialization of the three dimensional case, but owing to the difference in topology displays a remarkable simplicity and elegance of its own. Employing the results for harmonic fields we shall exhibit a number of relevant well-posed problems for the pre-Maxwell system and make some application of the methods to Faraday's Law, homopolar generators, and the pinch effect. In conclusion, there is a brief expository comparison between the Maxwell theory and the theory of the pre-Maxwell equations.

2. Topological Considerations

The classical integral theorems of Gauss and Stokes form the most natural avenue of approach to the formulation of well-posed boundary value problems for the systems (1.6)-(1.7). In each case the integral theorem expresses the value of an integral on a manifold in terms of an integral on the boundary, viz.[7]

$$\text{a.}\quad \int_C \operatorname{grad} \phi \cdot dx = \phi(x_2) - \phi(x_1)$$

$$(2.1)\qquad \text{b.}\quad \int_\Sigma \operatorname{curl} Z \cdot dS = \oint_\Gamma Z \cdot dx$$

$$\text{c.}\quad \int_D \operatorname{div} Z \, dV = \oint_S Z \cdot dS$$

The compatibility conditions mentioned in the introduction are restrictions upon the boundary values consistent with the given differential equations under (2.1). For example, if $\operatorname{div} Y = 0$ is one of the differential equations, then Gauss' formula (2.1c) implies that the normal component on the boundary, Y_n, must satisfy the condition $\oint Y_n dS = 0$. In the same way, Stokes' theorem places a restriction on the tangential boundary component X_t of a vector X which satisfies $\operatorname{curl} X = 0$. The compatibility requirements can be formulated precisely only in terms of topological properties of the given domain; specifically, in terms of homology.

---

[7] Here, and in the sequal, we systematically use the symbols, C for an open arc, Γ for a closed curve, Σ for an open surface, S for a closed surface, and D for a three-dimensional domain.

Homology of curves and surfaces. Relation to vacuum fields.

Let us take a three-dimensional domain D and consider, for example, the application of Stokes' Theorem (2.1b) to an arbitrary irrotational vector Z in D. If the surface $\sum$ which is bounded by the closed curve Γ lies entirely within the domain D, then $\oint_{\Gamma} Z \cdot dx = 0$. However, a closed curve Γ may exist which does not bound any surface $\sum$ lying entirely in D. In that event we may not conclude that $\oint_{\Gamma} Z \cdot dx = 0$ for every closed curve Γ merely because Z is irrotational. Nonetheless, to each Γ which does not bound in the domain D we may associate the class of all oriented closed curves Γ' such that $\Gamma - \Gamma'$ does bound a surface $\sum$ in D. Consider, for example, Γ and Γ'

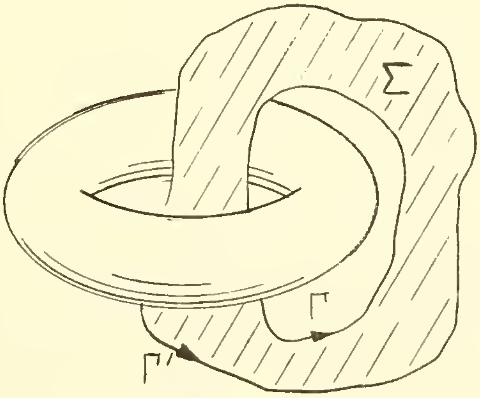

Figure 1. Closed curve homology in the exterior of a torus.

in Figure 1 where D is the exterior of a torus. It is clear for all curves of the class that $\oint_{\Gamma'} Z \cdot dx = \oint_{\Gamma} Z \cdot dx$. The value of the line integral therefore does not depend upon the curve Γ, but only upon this class, the homology class of Γ. Such a constant of a homology class is called a period of the vector field. The physical meaning of these results is clear. If the domain D is interpreted as a vacuum field external to a current carrying loop then the m.m.f. (magnetomotive force) $\oint_{\Gamma} B \cdot dx$ around a closed curve Γ in D represents the current in the loop linked by the given curve. This current is a period, clearly independent of the precise choice of the curve and dependent upon homology class only; in other words, the m.m.f. is the same for all closed paths which link the loop once in the same direction.

In an analogous fashion the divergence theorem will yield a concept of homology for surfaces when applied to solenoidal vectors. Consider, for example, a vacuum domain D between two concentric spheres (Figure 2). Two surfaces S and S' belong to the same homology class if $S - S'$ bounds a subdomain of D. In this case, all spherical surfaces concentric with the inner sphere belong to the same class. Clearly the divergence theorem implies that

$$\oint_{S'} Z \cdot dS = \oint_{S} Z \cdot dS \quad ,$$

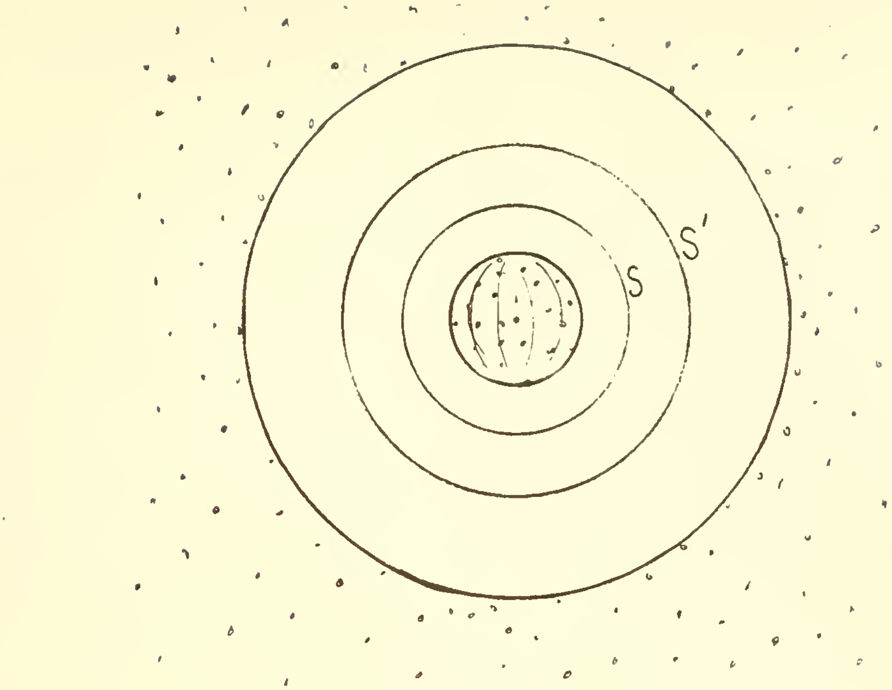

Figure 2. Closed surface homology between two spheres.

where S' and S lie in the same homology class. In this example, the electric flux through any sphere in D containing the excluded inner sphere is a period and is proportional to the total charge exterior to D within the inner sphere.

Incompressible fluid flow yields another example of a solenoidal vector field. The period $\oint u \cdot dS$, where u is the velocity vector, is assigned to the homology class of a closed surface S and is a measure of the sources contained in S.

We now proceed to expand somewhat on these concepts. However, so brief a description of the concept of homology as the present one must necessarily be quite incomplete. The reader is referred to standard works on topology for a thorough treatment[8].

The homology groups.

The three-dimensional domain D is assumed to be orientable and to possess a boundary consisting of finitely many regular bounded closed surfaces[9]. In addition, we shall assume regularity on the part of all curves mentioned. For example, D may be the exterior or the interior of a torus, the region between two

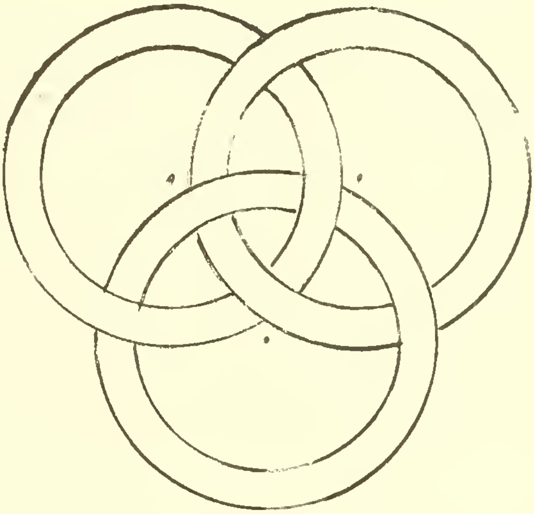

Figure 3. The Borromean rings.

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[8] Alexandroff, P. S., and Hopf, H., Topologie, Berlin, Springer, 1935.
Lefschetz, S., Introduction to Topology, Princeton University, 1949.
Seifert, H., and Threlfall, W., Lehrbuch der Topologie, B. G. Teubner, 1934.

[9] A domain is understood to be an open connected set. For the purpose of the integral theorems, piecewise smoothness is sufficient for regularity. For topological purposes it is required that the surface neighborhood of a point on the boundary is topologically equivalent to a disc. If the domain is infinite the sphere at infinity is to be thought of as part of the boundary, and in that case boundary conditions are replaced by regularity conditions. We shall not generally consider other infinite bounding surfaces.

concentric spheres, the exterior of the Borromean rings[10] (Figure 3), etc. In the following, we assume a finite topology, explicitly that there are only finitely many independent homology classes of each dimension. To explain this concept, let us recall the definition of addition of curves in integration theory which permits us to write

$$\int_{C_1+C_2} = \int_{C_1} + \int_{C_2}$$

(and similarly for surfaces). Each curve C has an additive inverse $-C$ obtained by reversing its orientation. Clearly, by the introduction of a zero element, the set of curves may be seen to form an additive group which contains the smaller additive group of closed curves.

A closed curve is said to be homologous to zero if it bounds a surface $\sum$ in D. The concept of bounding is

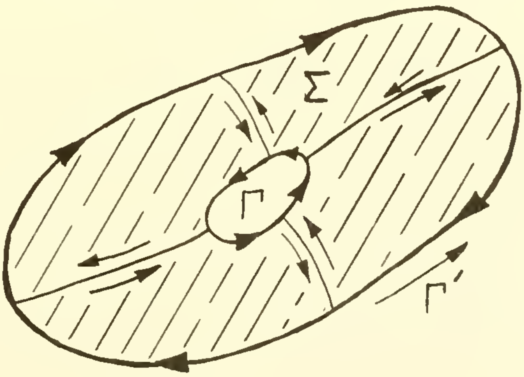

Figure 4. Boundary of an orientable surface.

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[10] Device on the coat-of-arms of the Milanese family of Borromeo (1406-), thought to be ancestors of the Ballentines.

roughly defined in the following way: A closed curve Γ is said to bound a surface $\sum$ if it constitutes the entire boundary of $\sum$ and if it is possible to partition $\sum$ into non-overlapping simple surface elements and orient their boundaries so that the sum of the bounding curves of the surface elements is the curve Γ. Thus, we see in Figure 4 that $\Gamma - \Gamma'$ bounds $\sum$ while $\Gamma + \Gamma'$ does not. In the domain D of Figure 1, the curve $\Gamma - \Gamma'$ is homologous to zero and we write $\Gamma - \Gamma' \sim 0$. Two curves Γ and Γ' are said to be homologous ($\Gamma \sim \Gamma'$) if their difference bounds. We may then speak of the addition of homology classes rather than curves, where the additive zero is simply the class of all curves which are homologous to zero. Integration over a linear combination of curves is defined by

$$\int_{\lambda C_1 + \mu C_2} Z \cdot dx = \lambda \int_{C_1} Z \cdot dx + \mu \int_{C_2} Z \cdot dx.$$

If curl $Z = 0$, then $\oint_{\Gamma} Z \cdot dx$ is seen by this formula to be a linear function on the linear vector space of homology classes of closed curves. This linear vector space is called the homology group of closed curves. The basis of the space is restricted to be finite. In the exterior of a torus (Figure 1), the curve Γ comprises the entire basis[11].

---

[11] We shall use any of the members of a homology class to designate that class.

The homology group of closed surfaces is defined in an entirely analogous way. A surface [illegible] is said to be homologous to zero if it bounds a subdomain of D. In the case of the interior of a torus, every closed surface bounds and the homology group is the null space.

Consider the example of the interior of a sphere with the interiors of two toruses deleted (Figure 5).

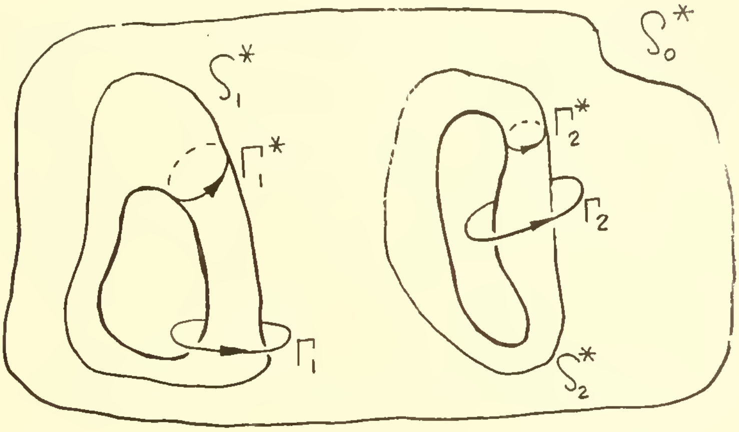

Figure 5. Homology in the interior of a sphere with two toruses deleted.

As a basis for the closed curve group, we may take the indicated curves Γ_1 and Γ_2. For the closed surface group, we may take the surfaces S_1^*, S_2^* of the toruses.

One further point of which we shall make significant use for the purpose of assigning compatible boundary values in the differential problems (1.6), (1.7): it is possible to take curves lying on the boundary as representatives of the homology classes. In Figure 5, for example, we may replace Γ_1 and Γ_2 by Γ_1^* and Γ_2^* on the boundary S^*[12]. We remark also that it is possible to consider the homology of closed curves with respect to the two-dimensional domain S^*; i.e., the criterion for homology is whether a curve Γ bounds a subdomain of S^*. We observe that closed curves which do not bound in D cannot bound on the two-dimensional boundary S^*. However, it is also possible to take representatives on S^* of the homology group of closed curves in the three-dimensional exterior $\overline{D}$ of D. In fact, it can be proved that the homology group of closed curves on S^* is the direct sum of the homology groups in D and $\overline{D}$.

The homology group of closed surfaces is more easily described than the group of closed curves. Let the boundary S^* of D consist of $q+1$ closed surfaces S_0^*, $S_1^*, \ldots, S_q^*$, oriented so that $S_0^* + S_1^* + \ldots + S_q^* \sim 0$, the sum bounding D. Since D may be considered as the domain common to the interior of one boundary component and the exterior of the others, clearly a basis for the homology group can be formed from any q of the components of the boundary.

It is convenient also to define the zero-dimensional homology. Two points are said to be homologous if they bound an arc in D. Homology of points in the domain D is of no importance since D is connected and every pair of points may be joined by an arc. The zero-dimensional homology will be significant here only for the exterior $\bar{D}$ to D, which has as many separated components as the common boundary S^*.

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[12] Starred manifolds will, without exception, refer to manifolds on the boundary of the domain D.

The analytical significance of homology.

Let Z be a regular vector function defined in D. To each closed curve and each closed surface S, we assign a real number by

$$(2.2)\qquad \begin{aligned} [\Gamma] &= [\Gamma;Z] = \oint_{\Gamma} Z \cdot dx \\ [S] &= [S;Z] = \oint_{S} Z \cdot dS \quad . \end{aligned}$$

If Y is solenoidal, div Y = 0, then, [S;Y] vanishes for any S which is homologous to zero. Consequently, the function [S;Y], which is defined on the class of closed surfaces, has the same value for each surface S in a given homology class. The constant associated in this way with a homology class (rather than with a particular surface) will be called a period. Similarly, if X is irrotational, curl X = 0 in D, then the function [C;X] is defined on homology classes independently of the choice of individual curves. Conversely, div Y = 0 (or curl X = 0) only if

$[S;Y]$ (or $[C;X]$) is constant on each homology class. It is true also that S_1 and S_2 (C_1 and C_2) are homologous if and only if $[S_1;Y] = [S_2;Y]$ ($[C_1;X] = [C_2;X]$) for every solenoidal Y (irrotational X). It would have been possible to define homology of two closed curves (surfaces) by requiring that the two integrals $[\Gamma;X]$ ($[S;Y]$) are the same for each irrotational X (solenoidal Y).

The condition $\operatorname{curl} X = 0$ guarantees the existence of a single-valued scalar potential locally. In order that it be possible to define a potential ϕ over all D with $X = \operatorname{grad} \phi$ it is both necessary and sufficient that the closed curve periods $[\Gamma;X]$ all vanish. Similarly, for any solenoidal vector Y a vector potential A exists locally with $Y = \operatorname{curl} A$. The necessary and sufficient condition that A be single valued in the whole domain D is that the closed surface periods $[S;Y]$ all vanish.

The topology has been assumed finite; that is, each closed curve or surface is homologous to a linear combination of a given finite set of basis elements. Let $\Gamma_1 \ldots \Gamma_p$ denote a basis of the closed curve group, $S_1 \ldots S_q$ a basis of the surface group. For every closed curve Γ and closed surface S we have the homology representations

$$(2.3)\quad \begin{cases} \Gamma \sim a_1\Gamma_1 + a_2\Gamma_2 + \dots + a_p\Gamma_p \\ S \sim b_1S_1 + b_2S_2 + \dots + b_qS_q \end{cases} .$$

When curl $X = 0$, div $Y = 0$, it follows that all periods are defined in terms of the finite set of periods of the basis elements by

$$(2.4)\quad \begin{cases} [\Gamma;X] = a_1[\Gamma_1;X] + \dots + a_p[\Gamma_p;X] \\ [S;Y] = b_1[S_1;Y] + \dots + b_q[S_q;Y] \end{cases} .$$

It is intuitively clear that an X or Y can always be found which will assign arbitrary given values to the set of periods of a basis. Furthermore, all that need be known about any solenoidal vector Y (or irrotational X) in order to determine $[S]$ (or $[C]$) for any closed surface (or curve) are the values of its periods on a basis.

Consider the example of two linked toruses bounded by surfaces S_1^* and S_2^* as indicated in Figure 6. The domain D is the exterior to both. The oriented closed curves Γ_1, $\hat{\Gamma}_1$, Γ_2, $\hat{\Gamma}_2$ are chosen in the surfaces S_1^* and S_2^* and Γ_3 links neither S_1^* nor S_2^*. Clearly $\Gamma_3 \sim 0$ while no one of Γ_1, $\hat{\Gamma}_1$, Γ_2, $\hat{\Gamma}_2$, separately, bounds in D. However $\Gamma_1 \sim \hat{\Gamma}_2$ and $\Gamma_2 \sim \hat{\Gamma}_1$ since each pair bounds an annular disc. Clearly Γ_1 and Γ_2 could be taken as a basis of the

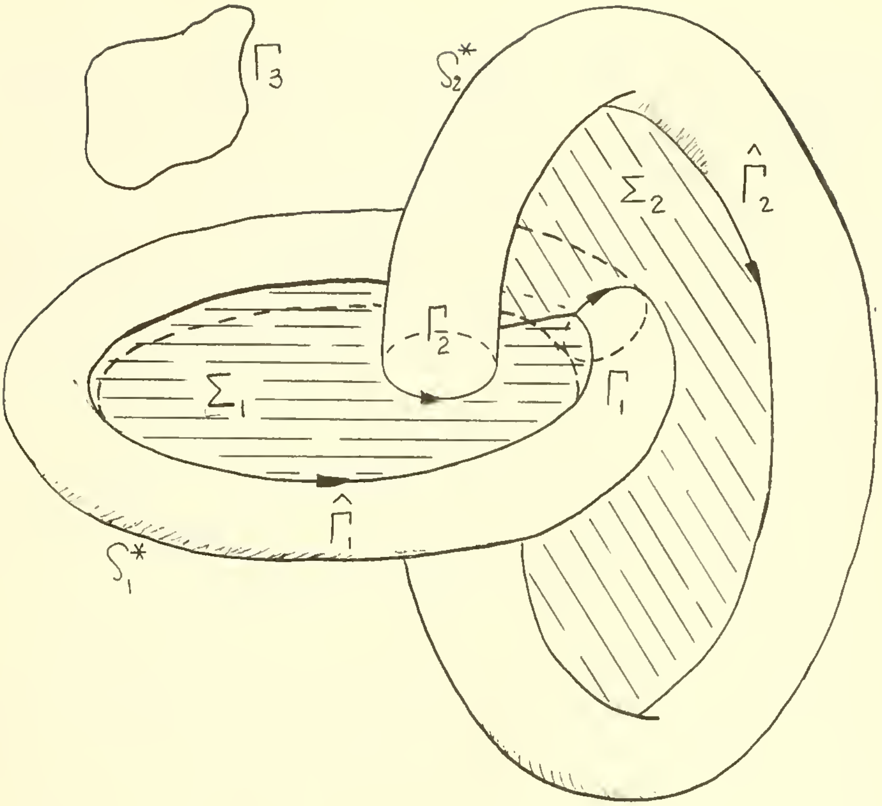

Figure 6. Homology in the exterior of two linked toruses.

closed curve group in **D**. There exists a two-parameter infinity of homology classes constructed so that every closed curve in D is homologous to $m\Gamma_1 + n\Gamma_2$ for some m and n. In this example, the closed surface group has a basis consisting of two elements which may be taken as S_1^* and S_2^*. (The third component of the boundary is the sphere at infinity). Note that homology alone does not distinguish between this case and that of Figure 5 although the two are topologically distinct.

3. Homology Modulo the Boundary.

The homology of closed manifolds has just been seen to be intimately connected with certain differential operators: the closed curve homology with the curl, the closed surface homology with the divergence. For an irrotational vector X we have observed that the integral about a closed curve Γ, $[\Gamma;X] = \oint_\Gamma X \cdot dx$, depends only on the homology class of Γ; for a solenoidal Y, the integral over a closed surface S, $[S;Y] = \oint_S Y \cdot dS$, again depends only on homology class. We shall find that when the homogeneous differential relation is supplemented with a suitable homogeneous boundary condition the relevant periods all vanish but other integrals arise which again depend only on the topology and not on the particular choice of the manifold of integration.

Consider, for example, a domain D and an irrotational vector X in D with $X_t = 0$ prescribed on the boundary S^*. Since a basis for the homology group of closed curves in D can be chosen on S^*, it is clear at once that $[\Gamma;X] = 0$ for all closed curves in D. Consider, now, two arcs C and C' which begin and terminate on S^*. If it is possible to find arcs on the boundary which taken together with C and C' form a closed curve Γ, then clearly, under the assumption $X_t = 0$, the only contributions made to $[\Gamma;X]$ come from C and C'. It follows that the integrals on C and C' are equal; the open arcs C' which with C can be completed to a closed curve by additions on the boundary constitute a class upon which the line integral $\int_{C'} X \cdot dx$ is constant.

This suggests a second kind of homology which we shall call homology modulo the boundary. This new concept of homology is most clearly understood by relating the homology of the domain D to that of the exterior domain $\bar{D}$ complementary to D.

The Alexander duality theorem.

There is a duality relation between homology in D and homology in the exterior domain $\bar{D}$ which is given by a remarkable general theorem of Alexander. For a three dimensional space, in particular, the Alexander duality theorem states that the number of independent elements of the k-dimensional homology group of D is equal to that of the $(3 - k - 1)$-dimensional group of $\bar{D}$. The two groups are called dual to each other.

From the duality theorem, it follows that the closed curve group in D is dual to the closed curve group in $\bar{D}$. As we have remarked, the homology group of closed curves on the common boundary S^* is the direct sum of the groups in D and the exterior $\bar{D}$. We conclude that the number of linearly independent closed curves on S^*

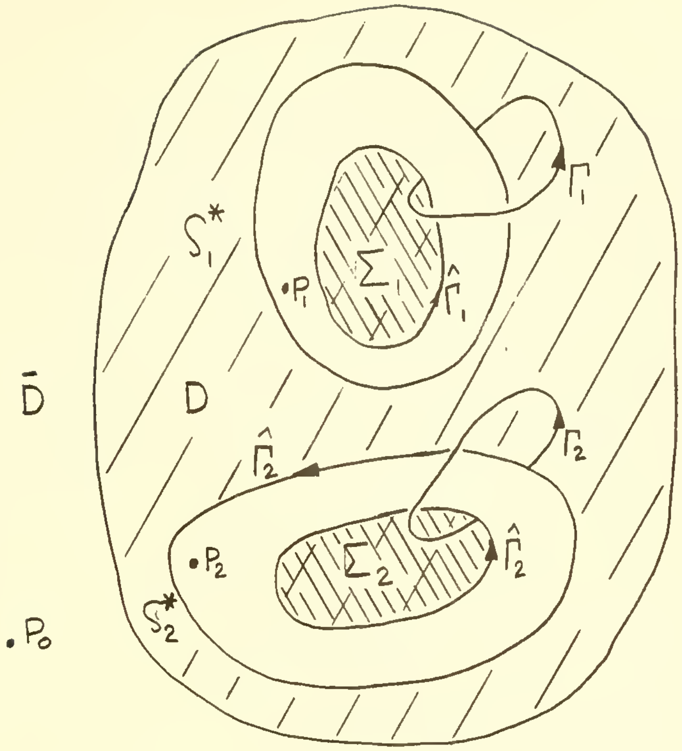

Figure 7. Dual bases in a sphere with two toruses deleted.

is double the number of those in D. For example, if D is the interior of a sphere with two toruses deleted depicted in Figure 7, Γ_1 and Γ_2 constitute a basis for D, and $\hat{\Gamma}_1$ and $\hat{\Gamma}_2$ for $\bar{D}$. The basis of the closed curve group on S^* consists of representatives from each of the four homology classes.

Homology of open curves modulo the boundary.

The dual to the homology of closed surfaces is easily visualized. Let the boundary S^* of D be written as the sum of its components, $S^* = S_0^* + S_1^* + \ldots + S_q^*$. To each component S_k^* we may assign a point P_k in the component of the exterior $\bar{D}$ bounded by S_k^* (Figure 7). The zero-dimensional homology group of $\bar{D}$ is generated by the basis P_0P_k, $(k = 1,2,\ldots,q)$.

If P_k and P_0 are taken on the boundary they may be connected by an arc C_k in D, (Figure 8).

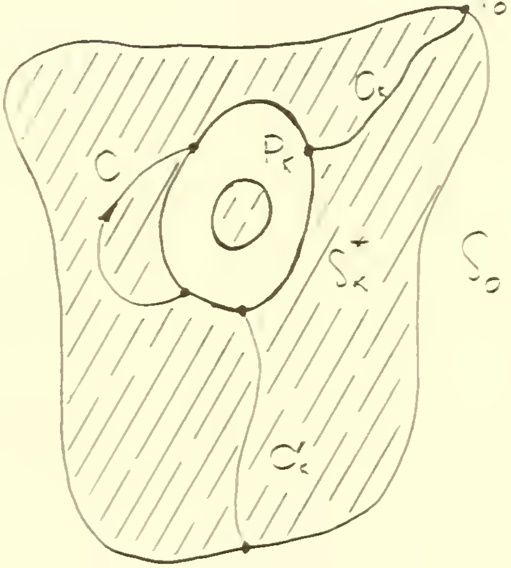

Figure 8. Representation of the open arc homology.

The arcs C_k may be taken as a basis for the homology in D of open curves modulo the boundary. Two curves C and C' in D which initiate and terminate on S^* are said to be homologous modulo the boundary ($C \sim C'$ [mod S^*]) if their zero-dimensional boundaries are homologous in $\bar{D}$. It is completely equivalent to set $C \sim C'$ [mod S^*] when it is possible to complete

$C - C'$ to a closed curve which bounds in D by adding arcs on the boundary only. The homology group of open arcs modulo the boundary in D is directly isomorphic to the zero-dimensional homology group in the exterior $\bar{D}$. Alexander's duality theorem therefore enables us to establish a duality between two homology groups in D, the group of closed surfaces $\{S\}$, and the group of open curves $\{C\}$.

The bases S^*_k and C_k, $(k = 1, \ldots, q)$, of the dual groups chosen above are *reciprocal*; i.e., the number of intersections (an intuitive concept which can be defined precisely) of C_j and S^*_k is equal to 1 if $j = k$ and equal to 0 otherwise.

Homology of open surfaces modulo the boundary.

The duality between the bases of the closed curve groups in D and $\bar{D}$ is somewhat more complex. It can be shown that there are dual bases, Γ_k in the domain D, and $\hat{\Gamma}_j$ in the exterior $\bar{D}$, $(j,k = 1, 2, \ldots, p)$ with the property that Γ_k links $\hat{\Gamma}_j$ exactly once if $j = k$ and not at all if $j \neq k$ (cf. Figure 7). Moreover, by taking the representative curves Γ_k and $\hat{\Gamma}_j$ on the boundary, S^*, we can obtain in D a duality between the closed curve homology and the *homology of open surfaces modulo the boundary*. The curves $\hat{\Gamma}_k$ do not bound in $\bar{D}$ or on S^*, but can be chosen to bound surfaces Σ_k in D. Each surface Σ_k will have the property that the intersection number of the closed curve Γ_j with the open surface Σ_k is 1 if $j = k$ and 0 otherwise. The bases Γ_j, Σ_k chosen in this way will then be reciprocal. The homology of the open surfaces Σ_k is defined above by the homology of the boundary closed curves $\hat{\Gamma}_k$ but it is intuitively clear that this is equivalent to setting $\Sigma_k - \Sigma'_k \sim 0$ [mod S^*] if it is possible to complete $\Sigma_k - \Sigma'_k$ to a bounding closed surface in D by adding a part of S^*.

An instructive example is that of the exterior domain to two linked toruses (Figure 6). The curves Γ_1, Γ_2 serve as a basis of the closed curve homology in D while $\hat{\Gamma}_1$ and $\hat{\Gamma}_2$ serve as a dual basis in $\bar{D}$. The closed curves $\hat{\Gamma}_1$ and $\hat{\Gamma}_2$ do not bound in D, but the curves $\hat{\Gamma}_1 - \Gamma_2$, $\hat{\Gamma}_2 - \Gamma_1$ in their respective homology classes bound annular domains, Σ_1, Σ_2. The open surfaces Σ_1, Σ_2 constitute a basis for the homology [mod S^*] which is reciprocal to the basis Γ_1, Γ_2.

The transformation rule for reciprocal bases is easily obtained. If Γ_j and Σ_k, $(j,k = 1,2,\ldots,p)$, are reciprocal bases and Γ'_j is a second basis for the closed curve group in D, then a reciprocal basis

$\sum_k'$, to Γ_j', is obtained from the $\sum_k$ by the inverse transpose of the linear transformation which takes the Γ_j into the Γ_j'. We shall restrict ourselves to reciprocal bases henceforth.

The analytical significance of homology modulo the boundary.

In the case of a solenoidal vector Y, we know that the surface integral $\oint_S Y_n \, dS$ is defined as a function of homology class alone and does not depend upon the special choice of the surface S. If the differential relation $\operatorname{div} Y = 0$ is supplemented by the condition that the normal component of Y vanish on the boundary, $Y_n = 0$ on S^*, then the periods $[S;Y]$ vanish on the basis S_k^*, $(k = 1,2,\ldots,q)$, chosen on S^* and consequently all closed surface periods vanish. In that event it becomes possible to define open surface periods in terms of the homology $[\text{mod } S^*]$. If $\sum$ is a surface in D with boundary on S^* we set

$$(3.1) \qquad [\textstyle\sum;Y] = \int_{\sum} Y \cdot dS \; .$$

The fact that the period $[\sum;Y]$ is a function of homology class alone is clear if we observe for any $\sum' \sim \sum$ $[\text{mod } S^*]$ that $\sum - \sum'$ can be completed to a closed bounding surface by the addition of part of S^*. The converse of the stated result is also true.

If the integrals $\int_{\Sigma} Y \cdot dS$ are periods, i.e., if they depend upon homology class alone, then the closed surface periods $[S;Y]$ vanish and $Y_n = 0$ on S^*.

These properties can also be stated interms of the vector potential. Since $\operatorname{div} Y = 0$ there exists a vector potential A, defined locally, for which $Y = \operatorname{curl} A$. If the open surface integrals are periods, there exists a single-valued vector potential A, where A is a local surface gradient on S^* [13]. This result is plain from

$$\oint_{\Gamma^*} A \cdot dx = \int_{\Sigma^*} \operatorname{curl} A \cdot dS = \int_{\Sigma^*} Y_n \, dS = 0 \quad ,$$

where Σ^* is any surface element on S^* and Γ^* is its boundary. If it should happen further that all periods $[\Sigma;Y]$ vanish then A is the surface gradient of a single-valued potential on S^*.

Similar results to the above can be obtained for an irrotational vector X. From $\operatorname{curl} X = 0$ it follows that the closed curve integral $\oint_{\Gamma} X \cdot dx$ is a period. If the boundary condition is that the tangential component vanish, $X_t = 0$ on S^*, then the periods $[\Gamma;X]$

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[13] For the purposes of this paper, a vector A is defined as a local surface gradient on the surface S if $\oint_{\Gamma} A \cdot dx = 0$ for any closed curve Γ bounding on S. We write $A = \nabla_S \phi$ where $\phi(x) = \int_{x_0}^{x} A \cdot dx$, the path of integration being taken in the surface.

vanish on the basis Γ_k^*, $(k = 1,2,\ldots,p)$, chosen on S^* and consequently all closed curve periods vanish. The open curve integrals for curves C with boundary on S^* now become periods,

$$(3.2) \qquad [C;X] = \int_C X \cdot dx \quad .$$

Conversely, if the integrals $\int_C X \cdot dx$ are periods then the closed curve periods $[\Gamma;X]$ vanish and $X_t = 0$ on S^*.

Since curl $X = 0$ there exists a local scalar potential ϕ with $X = \operatorname{grad} \phi$. If the open curve integrals are periods then ϕ is single-valued in D and constant on each component of S^*. If, in addition, all values $[C;X]$ vanish, then ϕ takes the same constant value on every component of S^*.

In each case there actually exists a vector which realizes arbitrarily specified period conditions on a basis. For example, if D is a vacuum domain, div $E = 0$. We may freely specify the charge $Q_k = \kappa[S_k^*,E]$, $(k = 1,2,\ldots,q)$, in the exterior domain to each of the elements of the basis of closed surfaces taken as components of the boundary with no restriction, but note that $\sum_0^q Q_k = 0$. Since div $B = 0$, the non-existence of magnetic charge implies that the periods $[S;B]$ always vanish. If we impose the condition $B_n = 0$ on S^* appropriate to a perfectly conducting boundary we observe that the open-surface periods are interpretable as magnetic fluxes. These fluxes $\Phi_k = [\Sigma_k ; B]$, $(k = 1,2,\ldots,p)$, may be given arbitrarily by specifying currents in external circuits $\hat{\Gamma}_k$ lying in the homology classes of the boundaries of the respective Σ_k.

For the purpose of convenient reference, the results concerning the relations between analytical and topological concepts are summarized in the following table. Periods are said to be well-defined if the integrals are determined by homology class alone.

Table I

Summary of the Relations of Analytical Concepts to Homology

Ia

i	curl $X = 0$	$[\Gamma;X]$ well-defined	
ii		$[\Gamma;X] = 0$	$X = \text{grad } \phi$
iii	curl $X = 0$, $X_t = 0$	$[C;X]$ well-defined	$X = \text{grad } \phi$, $\phi = \phi_k$ on S_k^*
iv		$[C;X] = 0$	$X = \text{grad } \phi$, $\phi = \text{const.}$ on S^*

Ib

i	div $Y = 0$	$[S;Y]$ well-defined	
ii		$[S;Y] = 0$	$Y = \text{curl } A$
iii	div $Y = 0$, $Y_n = 0$	$[\Sigma;Y]$ well-defined	$Y = \text{curl } A$; A is local surface gradient on S^*
iv		$[\Sigma;Y] = 0$	$Y = \text{curl } A$; $A = \nabla_S \phi$, ϕ single-valued on S^*

The table is to be read as follows. In each row of the table, the separate entries are equivalent. Row i in Ia, for example, should be read, "Closed curve periods $[\Gamma;X]$ are defined as functions of homology class alone, if and only if curl $X = 0$ in D." As one reads down a column of either table the conditions become increasingly more stringent so that each of the entries in a row implies all the entries on the rows above: so, for example, from $X = \operatorname{grad} \phi$ in Ia, ii we may conclude that $\operatorname{curl} X = 0$ and $[\Gamma;X]$ well-defined.

4. Compatibility Conditions.

The topological concepts developed in the two sections preceding will be reflected directly in the formulation of the boundary value problems of direct interest to us. From the analytical properties of homology we shall derive certain evident compatibility conditions which are to be employed in the formulation of well-posed problems for the pre-Maxwell equations.

Compatibility for values of the tangential boundary component of an irrotational vector.

Let us first consider an irrotational vector X, $\operatorname{curl} X = 0$ in the domain D and see what restrictions are imposed on the assignment of tangential boundary values X_t on S^*. For any closed curve Γ^* which bounds a surface element Σ^* on S^*, we must have $\oint_{\Gamma^*} X_t \cdot dx = \int_{\Sigma^*} \operatorname{curl} X \cdot dS$; hence,

$$(4.1) \qquad \oint_{\Gamma^*} X_t \cdot dx = 0 \qquad\qquad (\Gamma \sim 0 \text{ on } S^*).$$

If the line integral of the surface vector X_t vanishes for every curve Γ^* which bounds on S^*, then X is a local surface gradient (cf. Footnote, p. 41). On the boundary S^*, the homology group of closed curves has as a basis the $2p$ curves $\Gamma_1^* \ldots, \Gamma_p^*$; $\hat{\Gamma}_1^* \ldots \hat{\Gamma}_p^*$, where $\Gamma_1^* \ldots \Gamma_p^*$ is a basis for the homology group of closed curves in D, and $\hat{\Gamma}_1^* \ldots \hat{\Gamma}_p^*$ bound surfaces

$\Sigma_1, \dots \Sigma_p$ in D, which constitute a reciprocal basis for the homology group of open surfaces. It follows that

$$[\hat{\Gamma}_k^*, X] = \oint_{\hat{\Gamma}_k^*} X \cdot dx = \oint_{\hat{\Gamma}_k^*} X_t \cdot dx = \int_{\Sigma_k} \text{curl } X \cdot dS = 0 .$$

We see, then, that boundary values X_t cannot be assigned freely to an irrotational X in D, but that half the closed curve periods of X_t on the boundary S^* must vanish:

$$[\hat{\Gamma}_k^*, X] = \oint_{\hat{\Gamma}_k} X_t \cdot dx = 0 \qquad (k = 1, \dots, p) . \tag{4.2}$$

For an irrotational X we conclude that to prescribe the tangential field X_t compatibly on S^*, the tangential field must be a local surface gradient and half its periods must vanish while the remainder are completely determined by the given boundary values.

As an example of these restrictions, consider a helical

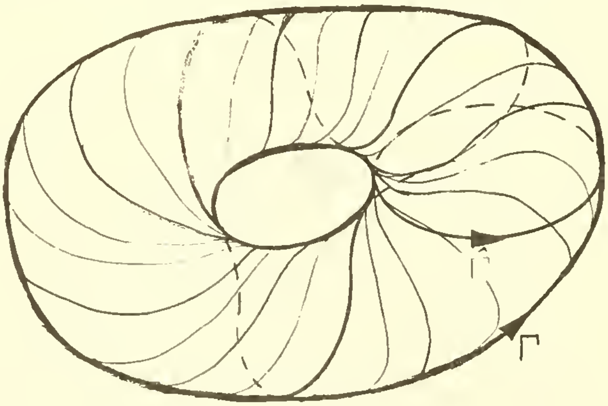

Figure 9. Helical vector field on the surface of a torus.

field X_t on the surface S^* of a torus (Figure 9). Suppose X_t to be a local surface gradient. There are two fundamental closed curves on S^* indicated in the figure by Γ and $\hat{\Gamma}$ where $\hat{\Gamma}$ bounds in the interior domain and Γ bounds in the exterior. To be compatible with irrotational X inside the torus, the transverse period $[\hat{\Gamma},X]$ must vanish (the pitch of the helix is said to be infinite). To be compatible with irrotational X in the exterior of the torus, the longitudinal period $[\Gamma,X]$ must vanish (the pitch of the helix is said to be zero).

Next consider an irrotational vector satisfying the homogeneous boundary condition $X_t = 0$. All closed curve periods vanish and the compatibility conditions are trivially satisfied. This time it is the open curves which become of interest. Open curve periods are defined by setting

$$(4.3) \qquad [C;X] = \int_C X \cdot dx$$

where C is an arc with endpoints on the boundary S^*. In that case, $X = \operatorname{grad} \phi$, where $\phi = \phi_k$, a constant, on the boundary component S_k^*. The periods on a basis are the potential differences

$$(4.4) \qquad \phi_k - \phi_0 = [C_k;X]$$

where C_k is an arc connecting S_k^* to S_0^*, $(k = 1,2,\ldots,q)$.

The conditions cited are indeed compatible in the sense that there always exists at least one vector X having boundary values X_t specified arbitrarily subject to the conditions (4.1) - (4.2). Moreover, in case $X_t = 0$, there exist vectors X for each possible specification of the constant values of the potentials on the boundary surfaces S_k^*, $(k = 0,1,2,\ldots,q)$.

Comptability for values of the normal boundary component of a solenoidal vector.

For a solenoidal vector Y, $\operatorname{div} Y = 0$ in D, the specification of compatible normal boundary values Y_n is simple. Let the boundary S^* be written again in the form $S^* = S_0^* + S_1^* + \ldots + S_q^*$ where the terms are the separate components of the boundary. Since $S^* \sim 0$ in D, values of Y_n will be assigned compatibly only if

$$(4.5) \qquad \oint_{S^*} Y_n \, dS = \sum_{k=0}^{q} [S_k^*; Y] = 0 \quad .$$

Under the homogeneous boundary condition $Y_n = 0$, all closed surface periods of the solenoidal field vanish, but now the open surface periods are defined in terms of homology modulo the boundary. If Σ is a surface in D with boundary on S^*, we set

$$(4.6) \qquad [\Sigma; Y] = \int_{\Sigma} Y \cdot dS \quad .$$

As in the preceding case, the conditions are compatible in the sense that there always exists a solenoidal Y for every specification of Y_n subject to [4.5]. In addition, in the case $Y_n = 0$, the open surface periods (4.6) may freely be assigned on a basis $\sum_1, \dots, \sum_p$.

Compatibility conditions for the boundary value problems with inhomogeneous differential relations.

If the differential equation is inhomogeneous, $\operatorname{curl} X = \Xi$ (with $\operatorname{div} \Xi = 0$) or $\operatorname{div} Y = \sigma$, then the concept of period as defined in terms of homology classes must be modified and the associated compatibility conditions altered accordingly.

In the case of the equation

$$\operatorname{div} Y = \sigma \tag{4.7}$$

where σ is a prescribed scalar function in D we have instead of (4.5) the compatibility condition,

$$\oint_{S^*} Y_n \, dS = \int_D \sigma \, dV \quad . \tag{4.8}$$

Furthermore, the integral $\oint_S Y \cdot dS$ is not constant on the homology class of S. Nonetheless, if $S' \sim S$ it is possible to evaluate the surface integral on S' in terms of that on S and the values of σ; for letting D' denote the domain bounded by $S' - S$ we have

$$\oint_{S'} Y \cdot dS - \oint_S Y \cdot dS = \int_{D'} \sigma \, dV \quad .$$

Therefore we set $[S;Y] = \oint_S Y \cdot dS$ and call the integral a period, but it must be kept in mind that the values of the integral have to be calculated from its value on a specified member of each homology class and from the known values of σ. In prescribing periods for an inhomogeneous differential system we must exert care to take fixed representatives in each homology class.

In exactly the same way we consider the equation

$$(4.9) \qquad \operatorname{curl} X = \Xi \; , \qquad\qquad (\operatorname{div} \Xi = 0) \; ,$$

where Ξ is a prescribed vector in D. The values of $\oint_\Gamma X \cdot dx$ do not depend upon homology class alone but depend also upon the values of Ξ. Thus, if $\Gamma' \sim \Gamma$ then $\Gamma' - \Gamma$ bounds a surface Σ in D and

$$\oint_{\Gamma'} X \cdot dx - \oint_{\Gamma} X \cdot dx = \int_{\Sigma} \Xi \cdot dS \; .$$

The definition of closed curve period must therefore be extended in terms of the values of Ξ. Again, specific values of the periods $[\Gamma;X] = \oint_\Gamma X \cdot dx$ must be assigned to fixed representatives of the homology classes, but when one such value is assigned all others in the same homology class are also determined.

The compatibility conditions on the boundary values X_t are also altered. No longer must X_t be a local surface gradient, but must satisfy instead of (4.1),

$$(4.10) \qquad \oint_{\Gamma^*} X_t \cdot dx = \int_{\Sigma^*} \Xi \cdot dS$$

where Σ^* is an arbitrary surface domain on S^* with boundary Γ^*. This relation amounts roughly to the specification of the surface curl of X_t as a given function on the surface; it determines X_t uniquely to within an arbitrary additive surface gradient. Furthermore, half of the closed curve periods on S^* are determined by the values of Ξ ; thus, instead of (4.2) we have

$$(4.11) \qquad \oint_{\hat{\Gamma}^*_k} X_t \cdot dx = \int_{\Sigma_k} \Xi \cdot dS \quad .$$

The period $[\hat{\Gamma}^*_k; X]$ will now depend upon the choice of the representative of the homology class. (However, the value of the period is independent of the particular surface Σ_k bounded by $\hat{\Gamma}^*_k$ since $\operatorname{div} \Xi = 0$.) These rather complicated compatibility conditions may be described conveniently in the following way. Let X^0 be any vector which satisfies $\operatorname{curl} X^0 = \Xi$. The vector X^0 has certain boundary values X^0_t on S^* and certain periods $\oint_{\hat{\Gamma}^*_k} X^0_t \cdot dx$ on the specified curves $\hat{\Gamma}^*_k$. For boundary values X_t to be compatible with the given values Ξ , X_t must differ from X^0_t by a local surface gradient and X_t must have periods equal to those of X^0_t on the $\hat{\Gamma}^*_k$.

The concept of open period for inhomogeneous boundary conditions.

The same type of generalization may be used to define periods on open curves and surfaces when the specified boundary values are inhomogeneous. For example, consider curl $X = 0$, X_t given; although $\int_C X_t \cdot dx$ is not a function of homology class [mod S^*] alone, the values of the integral for two curves in the same homology class differ by the value of a curvilinear integral on S^* which is known from the values of X_t. Similarly, if div $Y = 0$, Y_n given, the surface integrals on two open surfaces in the same homology class differ by an integral over a surface domain on S^* which is known in terms of the given values Y_n.

If the differential relations are inhomogeneous also, the difference between the open periods of two elements in the same homology class will have an additional term defined as above by the given data in D.

5. Existence Theorems for Harmonic Vectors and Related Fields.

In this section we formulate a number of well-posed problems for harmonic vectors and the related vectors satisfying inhomogeneous differential equations. The topology of the domain will be reflected in the existence theorems by the period conditions and restrictions on the boundary conditions.

The linear decomposition of solutions.

The solution of the pre-Maxwell equations can be reduced as we have indicated in the introduction to the successive solution of systems of the kind

$$(5.1) \qquad \begin{cases} \operatorname{div} Z = \sigma \\ \operatorname{curl} Z = \Xi \end{cases} \qquad (\operatorname{div} \Xi = 0) \, .$$

It is convenient to consider a solution of system (5.1) as the sum of solutions of the systems

$$(5.2) \qquad \begin{cases} \operatorname{div} Z = 0 \\ \operatorname{curl} Z = 0 \end{cases}$$

$$(5.3) \qquad \begin{cases} \operatorname{div} Z = \sigma \\ \operatorname{curl} Z = 0 \end{cases}$$

$$(5.4) \qquad \begin{cases} \operatorname{div} Z = 0 \\ \operatorname{curl} Z = \Xi \, , \end{cases} \qquad (\operatorname{div} \Xi = 0) \, .$$

The system (5.3) has the particular solution

$$\begin{aligned} &Z = \operatorname{grad} \phi \\ (5.5) \qquad &\phi = -\frac{1}{4\pi} \int_D \frac{\sigma}{r}\, dV \end{aligned}$$

and (5.4) has the particular solution

$$\begin{aligned} &Z = \operatorname{curl} A \quad , \\ (5.6) \qquad &A = \frac{1}{4\pi} \int_D \frac{1}{r}\, \Xi \, dV \quad . \end{aligned}$$

If H represents a harmonic vector, that is, a solution of (5.2), we see that a solution of (5.1) may be put in the form

$$(5.7) \qquad Z = H + \operatorname{grad} \phi + \operatorname{curl} A \quad .$$

Since ϕ and A are explicit, the role of properly posed boundary conditions is to fix H uniquely.

On the other hand, any given differentiable vector Z may be expressed in the form of the decomposition (5.7) by taking for σ and Ξ in (5.5) and (5.6), div Z and curl Z, respectively. We conclude that any vector Z can be written as the sum of a harmonic vector, a gradient, and a curl. A slightly different formulation is that every vector can be written as the sum of an irrotational and a solenoidal vector (H is both). In general, neither the irrotational nor the solenoidal vector has a single-valued potential; the special feature of (5.7) is that the multi-valuedness, if any, is concentrated in H. It is possible to refine such decompositions in such a way as to include in the statements themselves existence and uniqueness theorems of the systems (5.1) - (5.4). This will be done in the next section. But we shall first develop such existence theorems directly in the language of partial differential equations and in the next section restate them in terms of projections in a Hilbert space, a form particularly appropriate for variational analysis.

Existence theorems for harmonic vectors.

We first consider harmonic vectors, i.e., solutions Z, of (5.2) in a given domain D. The subtleties are purely topological. In a simply connected domain, we have $Z = \nabla \phi$ where $\Delta \phi = 0$. The problem for harmonic Z reduces to the scalar potential equation which can be solved, for example, if ϕ or $\partial\phi/\partial n$ (subject to $\oint \frac{\partial \phi}{\partial n} dS = 0$) is given on the boundary. For boundary conditions on Z we may prescribe either the tangential component Z_t as an arbitrary surface gradient or the normal component Z_n subject only to $\oint Z_n \, dS = 0$. In a more general domain the problem becomes one of specifying periods as well as boundary data, and verifying their compatibility. From the results of the last section, a harmonic Z always has well-defined periods

$\oint Z \cdot dx = [\Gamma]$ as well as $\oint Z \cdot dS = [S]$. With the boundary condition $Z_n = 0$, the periods $[\Sigma] = \int Z \cdot dS$ become meaningful while the periods $[S]$ vanish. If $Z_t = 0$, then the periods $[C] = \int Z \cdot dx$ become meaningful while the periods $[\Gamma]$ vanish. Note that specific periods $[\Gamma]$ and $[\Sigma]$ (both defined when $Z_n = 0$) are not, in general, equal in value even though there may be a correspondence between them in an isomorphism of the groups $\{\Sigma\}$ and $\{\Gamma\}$.

We have found four homology groups in D, closed surface denoted by $\{S\}$, open curve by $\{C\}$, closed curve $\{\Gamma\}$, and open surface $\{\Sigma\}$. By choosing reciprocal bases we have exhibited a duality between the closed curve group $\{\Gamma\}$ and the open surface group $\{\Sigma\}$, and another between the closed surface group $\{S\}$ and the open curve group $\{C\}$. In terms of linear spaces there are then only two abstract groups. Of the two abstract groups, the C,S group is the more often associated with the electric vector and will be denoted by $\mathcal{E}$. The periods of the $\mathcal{E}$ group in its two representations, open and closed, may be thought of as e.m.f.'s, $V = [C;E]$, or charges, $Q = K[S;E]$. The Γ,Σ group is usually more appropriate to the magnetic vector and is denoted by $\mathcal{M}$. The open and closed $\mathcal{M}$-periods are typically defined by fluxes $\Phi = [\Sigma;B]$, and currents $\mu I = [\hat{\Gamma};B]$ in external circuits. We shall on occasion refer to a set of periods of these abstract groups as $\mathcal{M}$-periods or $\mathcal{E}$-periods without specifying the particular representation of the abstract group.

The general well-posed problem will give boundary values of either Z_n and the values of a set of $\mathcal{M}$-periods or Z_t and a set of $\mathcal{E}$-periods. We shall see that the values of dual periods $[\Sigma]$ and $[\Gamma]$, for example, while not equal are related in the sense that specification of one, together with boundary conditions, determines the other and therefore only one set of periods for the $\mathcal{M}$ group may be fixed independently. A similar statement holds for the $\mathcal{E}$ group.

Without proof, we state[14]:

Theorem 1: There exists a unique harmonic vector Z with arbitrarily assigned $\mathcal{M}$-periods (in either the $\{\Sigma\}$ or $\{\Gamma\}$ representation) and with arbitrarily assigned values Z_n on the boundary S^* subject only to the compatibility condition (4.5), namely,

$$[S^*;Z] = \oint_{S^*} Z_n \, dS = 0$$

In particular, we have:

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[14] See preface for references.

Corollary: There exists a unique harmonic vector Z, with arbitrarily specified $\mathcal{M}$-periods for which $Z_n = 0$ on S^*.

The corollary is especially simple in not requiring the extended definition of the open surface periods; all periods are functions of homology class alone and compatibility is automatic.

As an example of the above corollary consider a vacuum magnetic field B in a domain bounded by conductors. We take $B_n = 0$ as the appropriate boundary condition. The magnetic vector B will be uniquely determined by a set of specified fluxes $\Phi_k = [\Sigma_k;B]$, or by the m.m.f.'s $[\Gamma_k;B]$, $k = (1,2,\ldots,p)$; where the Σ_k constitute a basis of the open surface group, and the Γ_k are a reciprocal basis of the closed curve group. If the Γ_k bound open surfaces $\hat{\Sigma}_k$ in the exterior $\bar{D}$ then the m.m.f.'s may be interpreted as current fluxes given by $\mu[\hat{\Sigma}_k;J]$ or as currents μI_k in the external circuits $\hat{\Gamma}_k$.

We are also interested in problems for which the tangential field component Z_t is prescribed on the boundary. Now it is the periods on a representation $\{S\}$ or $\{C\}$ of the $\mathcal{E}$ group which must be specified.

We state without proof:

Theorem 2: There exists a unique harmonic vector with arbitrarily assigned $\mathcal{E}$-periods (in either the $\{S\}$ or $\{C\}$ representation) and boundary values Z_t which are local surface gradients and satisfy the compatibility condition (4.2) on S^*, but are otherwise arbitrary. In particular, we have,

Corollary: There exists a unique harmonic vector with arbitrary assigned $\mathcal{E}$-periods (in either representation) and homogeneous boundary values $Z_t = 0$ on S^*. Again the corollary does not require the extended definition of the open periods.

As an illustration of the corollary consider a vacuum electrostatic field $\mathbf{E}$ in a domain D bounded by conductors. The correct boundary condition is $\mathbf{E}_t = 0$. The electric vector is determined uniquely either by a set of specified potential differences $\phi_k - \phi_0 = [C_k;\mathbf{E}]$, or by specified external charges $Q_k = \kappa[S_k^*;\mathbf{E}]$, $(k = 1,2,\ldots,q)$; where the C_k and S_k^* denote the reciprocal bases of the open curve and closed surface groups.

Existence theorems for the inhomogeneous differential systems.

Turning now to the inhomogeneous problems (5.3), (5.4), (5.1), we note from the fact that the solutions (5.5), (5.6) exist and determine the general solutions to within an added harmonic vector, that a meaningful problem may be expected to employ the same sort of boundary condition and period condition as for harmonic vectors. The only necessary caution is that it now becomes necessary to extend the definition of periods even on closed curves and surfaces to make allowance for the given values of σ and Ξ. The values of periods must be assigned to fixed representatives of homology classes rather than to the classes themselves.

In the light of the above observations, we have:

Theorem 3: The inhomogeneous systems (5.3), (5.4), (5.1) possess unique solutions under the conditions of Theorem 1 or Theorem 2 with the compatibility conditions referred to the prescribed fields σ and Ξ.

As a consequence of the above theorems, we shall prove a further theorem which will be useful in the sequel.

Theorem 4: There exists a unique solution of the system

$$(5.8)\qquad \begin{cases} \operatorname{curl}\operatorname{curl} \mathbf{A} = \Xi & \qquad (\operatorname{div} \Xi = 0) \\ \operatorname{div} \mathbf{A} = \sigma \end{cases}$$

with arbitrary boundary values $\mathbf{A}_t$ and with either arbitrarily prescribed periods $[\mathbf{C};\mathbf{A}]$ or prescribed periods $[\mathbf{S};\mathbf{A}]$ compatible with σ.

The accounting of boundary data to see whether they seem to be sufficient to specify a solution is fairly subtle. The equation curl curl $\mathbf{A} = \Xi$ is akin to a potential equation for each of the three components of A (it is exactly this if $\Xi = 0$ and $\sigma = 0$). However, the number of scalar boundary conditions is two (A_t is a two-component vector); this loss of one boundary condition is necessary in order to satisfy the other differential equation, $\operatorname{div} A = \sigma$. In the simpler case where only one "curl" is present, $\operatorname{curl} A = \Xi$, we may impose only a single scalar boundary condition, A_t = a surface gradient; the additional "curl" allows an extra boundary condition.

To demonstrate theorem 4, we set $\operatorname{curl} A = Z$ and solve in turn

$$(5.9) \qquad \begin{cases} \operatorname{curl} Z = \Xi, & (\operatorname{div} \Xi = 0) \\ \operatorname{div} Z = 0 \end{cases}$$

and

$$(5.10) \qquad \begin{cases} \operatorname{curl} A = Z, & (\operatorname{div} Z = 0) \\ \operatorname{div} A = \sigma \end{cases}$$

By differentiating A_t on the boundary, (or by computing $\lim \frac{1}{S} \oint A_t \cdot dx$) we compute the boundary values of Z_n, and obtain the periods $[\sum; Z]$ by integration. Since A_t is given as single-valued, these values are compatible. A unique solution of (5.9) therefore exists (compatibility with respect to Ξ has been taken care of). Now we turn to (5.10). The boundary values A_t are compatible with the inhomogeneous term $\mathbf{Z}$ by our construction of $\mathbf{Z}$ and the periods are assigned either in the form $[C;A]$ without restriction or in the form $[S;A]$ compatible with σ by hypothesis. Hence a unique solution of (5.10) exists, and the theorem is proved.

Theorem 4 has application to the system

$$(5.11) \qquad \begin{cases} \operatorname{curl}\operatorname{curl} E = -\mu \dot{J} \\ \operatorname{div} \mathbf{E} = q/\kappa \end{cases}$$

obtained by eliminating B from the pre-Maxwell equations. Here q and J are assumed to be given as functions of space and time. The boundary value E_t can be specified arbitrarily and also the total charge in the exterior domain of each boundary surface (with one exception) and this will determine the electric field uniquely. Roughly, a general field E satisfying (1.1) is distinguished from an electrostatic field E $(q = 0, \partial B/\partial t = 0)$ in that E_t is arbitrary rather than merely an arbitrary surface gradient. The specification of E fixes $\partial B/\partial t$, but not quite B; some initial values are required for this, and the matter will be discussed in Section 10.

6. Orthogonal Decompositions.

In working with the system (5.1), it is natural to utilize the customary viewpoint of linear analysis and to write a solution in terms of the decomposition (5.7) into harmonic, irrotational, and solenoidal vectors. It is possible, moreover, to make refinements in the decomposition so as to make the components orthogonal in the sense of Hilbert space projections.

Inner product formulas. (Hodges' Theorem.)

We introduce on the class of continuously differentiable vectors, the inner product[15]

$$(X,Y) = \int_D X \cdot Y \, dV \tag{6.1}$$

A simple expression for the inner product in terms of periods and boundary values alone may be obtained when one vector is irrotational and the other solenoidal,

$$\begin{cases} \operatorname{curl} X = 0 \\ \operatorname{div} Y = 0 \end{cases} \tag{6.2}$$

There exists a local potential such that $X = \operatorname{grad} \phi$. The local potential may be taken to be single valued by introducing suitable cross cuts in D. To describe the cuts let us take reciprocal bases $\Gamma_1, \ldots, \Gamma_p$ in $\{\Gamma\}$ and $\Sigma_1, \ldots, \Sigma_p$ in $\{\Sigma\}$, (i.e., Γ_j intersects Σ_j once and Σ_k, $k \neq j$, not at all). Evidently the potential ϕ can be made single-valued when we take the Σ_k as cuts in the domain D.

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[15] Note that we are not dealing with a complete Hilbert space.

From the identity

$$\text{(6.3)} \qquad \operatorname{div}(\phi Y) = \operatorname{grad}\phi \cdot Y + \phi \operatorname{div} Y$$

we have $X \cdot Y = \operatorname{div}(\phi Y) - \phi \operatorname{div} Y = \operatorname{div}(\phi Y)$ and upon integration over the cut domain

$$\int_D X \cdot Y \, dV = \oint_{S^*} \phi Y \cdot dS + \sum_{k=1}^{p} \int_{\Sigma_k} [\phi] Y \cdot dS$$

where $[\phi]$ is the jump in ϕ across Σ_k. Since $[\phi] = \oint_{\Gamma_k} X \cdot dx$ and $\operatorname{curl} X = 0$, it follows that $[\phi]$ is constant on Σ_k, $[\phi] = [\Gamma_k; X]$. Consequently,

$$\text{(6.4)} \qquad (X,Y) = \oint_{S^*} \phi Y \cdot dS + \sum_{k=1}^{p} [\Gamma_k; X]\,[\Sigma_k; Y] \quad .$$

The period notation in (6.4) is used without any implication that the values are constant on homology classes. The values are given for specific curves Γ_k and surfaces Σ_k. In particular, not only the periods but also the surface integral may depend on the choice of cuts. The integrals on the right depend upon the particular choice of the cuts Σ_k, but if we assume the boundary condition $Y_n = 0$ on S^*, we have (for irrotational X and solenoidal Y),

$$\text{(6.5)} \qquad (X,Y) = \sum_{k=1}^{p} [\Gamma_k; X]\,[\Sigma_k; Y], \qquad (Y_n = 0 \text{ on } S^*) \ .$$

If on the other hand $X_t = 0$ on S^*, then from Table Ia, iii, we have $\phi = \phi_k$, constant, on S_k^*. From $\phi_0 \sum_0^q [S_k^*; Y] = 0$, it follows that

$$(X,Y) = \sum_{k=0}^{q} \phi_k [S_k^*; Y] = \sum_{k=1}^{q} \phi_k [S_k^*; Y] - \sum_{k=1}^{q} \phi_0 [S_k^*; Y] \quad .$$

Since $\phi_k - \phi_0 = \int_{C_k} X \cdot dx$, we have, for X and Y satisfying (6.2),

$$(6.6) \quad (X,Y) = \sum_1^q [C_k;X]\,[S_k^*;Y] \,, \qquad (X_t = 0 \text{ on } S^*) \,.$$

The results (6.5) and (6.6) together constitute a special case of Hodge's theorem for the domain D.

In summary, for X irrotational and Y solenoidal the inner product (X,Y) depends on boundary values and periods alone (6.4). If, in addition, either X or Y satisfies an appropriate homogeneous boundary condition, the inner product is expressible as a bilinear form in the periods alone, for one case, in terms of $\mathcal{E}$-periods (6.5), for the other, in terms of $\mathcal{M}$-periods (6.6).

The inductance and capacitance matrices.

If Z is harmonic $Z = X = Y$ in (6.2) and satisfies either boundary condition $Z_n = 0$ or $Z_t = 0$, then the equations (6.5) and (6.6) express the Dirichlet Integral $\int_D Z^2\, dV$ in terms of periods alone. If the boundary condition $Z_n = 0$ is prescribed, then $\int_D Z^2\, dV$ is expressed as a bilinear form in the periods of the two representations of the $\mathcal{M}$ group; hence, by taking $Z = B$, the magnetic energy $\frac{1}{2\mu} \int_D B^2\, dV$ may be expressed as a bilinear form in currents and fluxes. Similarly, under the boundary condition $Z_t = 0$, the Dirichlet integral is expressed as a bilinear form in the periods of the two representations of the $\mathcal{E}$ group; taking $Z = E$, we have the electric energy $\frac{1}{2}\kappa\int_D E^2\, dV$ expressed in terms of charges and potential differences. In each of these cases the periods of one representation of the abstract group are related by a linear transformation to the periods of the other representation. In the first example the transformation is given by the inductance matrix and in the second, by the capacitance matrix. In either case, by employing the matrix it is possible to express the Dirichlet integral as a quadratic form in the periods of a single representation of the abstract group.

Specifically, we consider the case of a vacuum magnetic field in a domain D bounded by perfectly conducting surfaces upon with $B_n = 0$. Since the prescription of one class of $\mathcal{M}$-periods together with the condition $B_n = 0$ determines B uniquely, the other class of $\mathcal{M}$- periods is also determined. The closed curve periods $[\Gamma_k ; B]$ are determined by the currents I_k in the external circuits $\hat{\Gamma}_k$, where $[\Gamma_k ; B] = \mu\, I_k$. The associated fluxes are $\Phi_k = [\Sigma_k ; B]$. To exhibit the relation between the I_k and the Φ_k, we represent B in two ways as the sum of special solutions. Under the condition $B_n = 0$ let B_j be the unique solution of the problem[16]

$$(6.7a) \qquad [\Gamma_k ; B_j] = \mu\, \delta_{kj} \qquad (j = 1,\dots,p)$$

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[16] The Kronecker symbol δ_{kj} is defined by

$$\delta_{kj} = \begin{cases} 0\ , & k \neq j \\ 1\ , & k = j \end{cases}$$

and B_j' the unique solution for which

$$(6.7b) \qquad [\Sigma_k ; B_j'] = \delta_{kj} .$$

The field B_j is then the field obtained when unit current is applied in the one external circuit $\hat{\Gamma}_k$. Similarly, B_j' is the field for which unit flux exists across Σ_j but the flux is zero across every other independent open surface. For an arbitrary solution B specified either by its currents $I_1, I_2, \ldots, I_p$ or by its fluxes $\Phi_1, \Phi_2, \ldots, \Phi_p$, we may then put

$$(6.8) \qquad \begin{aligned} B &= I_1 B_1 + I_2 B_2 + \ldots + I_p B_p \\ &= \Phi_1 B_1' + \Phi_2 B_2' + \ldots + \Phi_p B_p' \end{aligned}$$

This is a specific exemplification of the statement that the linear space of harmonic vectors in D is finite dimensional. In particular, for each B_j we may calculate the fluxes and obtain

$$(6.9) \qquad B_j = \sum_{k=1}^{p} L_{jk} B_k' , \qquad (j = 1, \ldots, p) ,$$

where L_{jk} is defined by

$$(6.9a) \qquad L_{jk} = [\Sigma_k ; B_j] .$$

Hodge's Theorem (5.6) gives

$$(B_j;B_k) = [\Gamma_j;B_j]\,[\Sigma_j;B_k] = \mu L_{kj}$$

$$= [\Gamma_k;B_k]\,[\Sigma_k;B_j] = \mu L_{jk}$$

where by taking B_j and B_k in turn as **X** we see that the matrix L_{jk} is symmetric. From (6.8) and (6.9) we have $B = \sum_{j=1}^{p} \sum_{k=1}^{p} I_j L_{jk} B_k'$ and obtain the fluxes by

$$\Phi_k = \sum_{j=1}^{p} L_{jk}\, I_j \quad ; \tag{6.10}$$

we see then that L_{jk} is the classical inductance matrix.

The magnetic energy now may be given the simple representation,

$$\begin{aligned} \frac{1}{2} \int_{\mathbf{D}} \frac{B^2}{\mu}\, dV &= \frac{1}{2\mu} \sum_{k=1}^{p} [\Gamma_k;B]\,[\Sigma_k;B] \\ &= \frac{1}{2} \sum I_k\, \Phi_k = \sum I_k\, L_{jk}\, I_j \quad , \end{aligned} \tag{6.11}$$

from which it follows that the matrix is positive definite.

If the basis Γ, written as a vector, is transformed to a new basis Γ' by a matrix (M), then Σ is transformed to a reciprocal basis Σ' by a related transformation

$$\Gamma' = (M)\,\Gamma \,, \qquad \Sigma' = (M^*)^{-1}\,\Sigma$$

where $(M^*)^{-1}$ is the reciprocal transpose of M. Accordingly we have

$$\Phi' = (M^*)^{-1}\,\Phi = (M^*)^{-1}\,(L)\,I = (M^*)^{-1}(L)(M^{-1})\,I'$$

so that (L) transforms according to the rule

$$(L) \longrightarrow (M^{*})^{-1} (L) (M^{-1})$$

The induction matrix is uniquely determined by the domain D and by the choice of a basis for $\mathcal{M}$, that is a choice of external circuits carrying assigned currents I_k or of surfaces with assigned fluxes Φ_k. It is important to remember that to express the energy in the simple form (6.11) it is necessary to take reciprocal bases in defining the currents and fluxes.

In direct analogy with the preceding case there is a symmetric positive-definite capacitance matrix relating the potentials to the charges if the harmonic vector is E and the homogeneous condition $E_t = 0$ is imposed. The e.m.f.'s and charges are defined on reciprocal bases by $V_k = \phi_k - \phi_0 = [C_k; E]$ and $Q_k = K[S_k^{*}; E]$; $(k = 1,2,\ldots,q)$. These are related by the classical capacitance matrix

$$Q_i = \sum_{j=1}^{q} C_{ij} V_j \qquad (i = 1,2,\ldots,q) \tag{6.12}$$

where C_{ij} is defined in the manner of the preceding example. The electrostatic energy is given by

$$\frac{1}{2} K \int_D E^2 \, dV = \frac{K}{2} \sum_{k=1}^{q} [C_k;E]\,[S_k^*;E]$$

(6.12a)

$$= \frac{1}{2} \sum_{k=1}^{q} V_k Q_k = \sum V_k C_{kj} V_j \; .$$

Since we assume for the total charge of space $\sum_{k=0}^{q} Q_k = 0$ we may write (6.12a) in the symmetric form

(6.12b) $$\int_D \frac{1}{2} E^2 \, dV = \frac{1}{2} \sum_{k=0}^{q} \phi_k Q_k$$

Orthogonal decomposition of vectors.

Hodge's theorem yields a number of useful refinements of the decomposition (5.7) of an arbitrary vector into irrotational and solenoidal components. First we observe that X and Y are orthogonal, $(X,Y) = 0$, if either of the conditions in the following table is satisfied.

Table II

Orthogonality Conditions for Irrotational and Solenoidal Vectors, curl $X = 0$, div $Y = 0$.

a.	$X_t = 0$ on S^*. $[C;X]$ or $[S;Y] = 0$.
b.	$Y_n = 0$ on S^*. $[\Gamma;X]$ or $[\Sigma;Y] = 0$.

From Table IIb it follows that the harmonic vectors L with $L_n = 0$ are orthogonal to the harmonic vectors M with $[\Gamma;M] = 0$. From Table IIa harmonic vectors N with $N_t = 0$ are orthogonal to harmonic vectors M with $[S;M] = 0$. The three classes of harmonic vectors satisfying the conditions of the following table are, then, mutually orthogonal.

Table III

Orthogonality Conditions for Harmonic Vectors

L	M	N
$L_n = 0$	$[\Gamma;M] = [S;M] = 0$	$N_t = 0$

The value of Table III lies in the fact that the decomposition is complete as well as orthogonal. In other words, every harmonic vector may be assigned a unique representation as a sum of orthogonal harmonic vectors, one from each class. Let H be an arbitrary harmonic vector which takes on the boundary values H_n, H_t and has the periods $[\Gamma;H]$, $[S;H]$. From the existence theorems for harmonic vectors we can determine, in turn, the three unique vectors

$$L: \quad L_n = 0 \ , \quad [\Gamma;L] = [\Gamma;H]$$

$$N: \quad N_t = 0 \ , \quad [S;N] = [S;H]$$

$$M: \quad M = H - L - N \ .$$

Plainly, $H = L + M + N$ is the required decomposition.

In the same way, by applying the existence theorems for harmonic vectors we obtain from Table II an orthogonal refinement of the decomposition (5.7) of an arbitrary vector Z into irrotational, harmonic, and solenoidal components; namely, components satisfying the conditions of the following table.

Table IV

Orthogonality Conditions for Irrotational, Harmonic and Solenoidal Vectors.

X (=grad ϕ)	H	Y (=curl A)
curl $X = 0$	div $H = 0$	div $Y = 0$
$X_t = 0$	curl $H = 0$	$Y_n = 0$
$[C;X] = 0$		$[\sum;Y] = 0$

Again, this decomposition is not only orthogonal but also complete; *every vector Z* dan be uniquely expressed as the sum of three vectors, one from each column. The headings curl A and grad ϕ are indicative of the fact that single-valued potentials exist under the conditions stated. The middle column can, of course, be further subdivided as in Table III. This is the orthogonal refinement of (5.7) that was previously referred to.

To prove the completeness of Table IV, we start with the non-orthogonal decomposition (5.7),

$$Z = H_0 + \text{grad } \phi_0 + \text{curl } A_0 \ .$$

There exist unique harmonic vectors, H_1 with the same tangential boundary values and $\{C\}$ periods as grad ϕ_0, and H_2 with the same normal boundary values and $\{\Sigma\}$ periods as curl A_0. The period and boundary data are clearly compatable. Setting $H = H_0 + H_1 + H_2$, grad ϕ = grad $\phi_0 - H_1$, curl A = curl $A_0 - H_2$, we obtain the decomposition in the desired form.

It will be convenient also to have this result in the form of

Theorem 5: Any arbitrary vector Z can be written in the form

$$Z = \text{curl } A + H + \text{grad } \phi \tag{6.13}$$

where $\phi = 0$ and $A_t = 0$ on the boundary.

Theorem 5 follows from the fact that the first column in IV describes the gradient of a potential function which is constant on S^* and may, therefore, be taken as zero there. At the same time, the third column in IV describes the curl of a vector potential A where A_t is a surface gradient and

$$\int_{\hat{\Gamma}_k} A_t \cdot dx = 0 \qquad (k = 1,\ldots,p) \ .$$

Consequently, by subtracting a suitable harmonic vector from A, we can obtain an A' for which $A'_t = 0$.

We have used the existence theorems of the last section to prove completeness of the orthogonal decompositions of Tables III and IV. A more common procedure is the reverse; namely, to enlarge the space of differentiable vectors X and Y to obtain a complete Hilbert space, appropriately generalizing the differential relations (6.2) and the boundary conditions so that they do not imply smoothness. The existence and uniqueness of the orthogonal decompositions then follow by application of the projection theorem in Hilbert space. A proof in this way of the existence theorems of the last section is trivial in the class of Hilbert space elements, but additional effort is required to verify actual differentiability of the harmonic vectors[17].

Further inner product formulas.

Since we shall be dealing with inhomogeneous systems, it is not sufficient to work with the representations of inner products given by (6.5) and (6.6). For the system inhomogeneous in Y

$$\begin{aligned} \operatorname{curl} X &= 0 \\ \operatorname{div} Y &= \sigma \end{aligned} \tag{6.14}$$

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[17] Courant-Hilbert, Methoden der Mathematischen Physik, Berlin, Springer (1937), Vol. II, Chap. VII.

equation (6.3) yields

$$(6.15)\quad (X,Y) = -\int_D \phi\,\sigma\,dV + \int_{S^*} \phi Y \cdot dS + \sum_{k=1}^{p} [\Gamma_k ; X][\Sigma_k ; Y].$$

For the system inhomogeneous in X,

$$(6.16)\quad \begin{aligned} \operatorname{curl} X &= \Xi \qquad\qquad (\operatorname{div} \Xi = 0) \\ \operatorname{div} Y &= 0 \quad , \end{aligned}$$

matters are somewhat more complicated.

To obtain a single-valued vector potential A with $Y = \operatorname{curl} A$, cuts are made along the open arcs $C_1,\dots,C_q$. To perform the necessary integrations, the cuts C_k are enclosed in narrow tubes T_k $(k = 1,\dots,q)$, Figure 10. Upon integrating the identity

$$(6.17)\quad \operatorname{div}(A \times X) = X \cdot \operatorname{curl} A - A \cdot \operatorname{curl} X$$

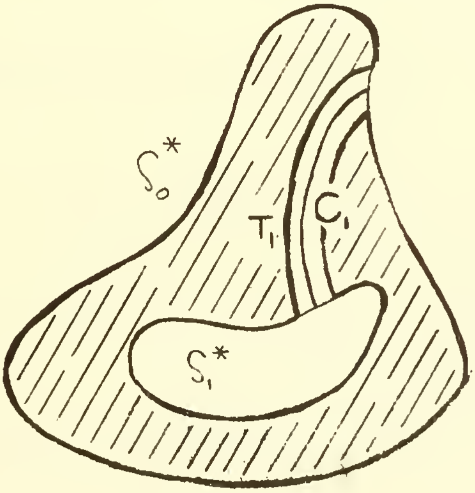

Figure 10. A domain cut by an open arc C, for the purpose of obtaining a single-valued vector potential.

we obtain, excepting small elements,

$$(X,Y) = (A,\Xi) + \int_{S^*} A \times X \cdot dS + \sum_{k=1}^{q} \int_{T_k} A \times X \cdot dS$$

First we note that

$$\int A \times X \cdot dS = \int A_t \times X_t \cdot dS \quad .$$

If T_k is permitted to shrink to C_k, the integral of A around the tube approaches the constant $[S_k^*;Y]$ (by Stokes' Theorem), and since X approaches a value depending only on the limiting position on C_1, we find

$$\int_{T_k} A \times X \cdot dS \longrightarrow [S_k^*;Y]\ [C_k;X]$$

We note that $\int A \times X \cdot dS$ converges since the singularity in A_t is integrable. Finally we have

$$(6.18) \quad (X,Y) = (A,\Xi) + \int_{S^*} A_t \times X_t \cdot dS + \sum_{k=1}^{q} [S_k^*;Y][C_k;X]$$

Each of the terms on the right in (6.15) and (6.18) depends on the choice of cuts. The field A may be singular at a cut and ϕ cannot in general be extended in single-valued fashion across cuts.

If $\sigma = 0$ and $\Xi = 0$, (6.15) and (6.18) combined yield a relation between surface fields and periods, namely,

$$\int_{S^*} A_t \times X_t \cdot dS - \int_{S^*} \phi Y \cdot dS = \sum_{k=1}^{p} [\Gamma_k ; X][\Sigma_k ; Y] -$$

$$- \sum_{k=1}^{q} [S_k^* ; Y] \, [C_k ; X]$$

From this we can obtain a useful identity involving two arbitrary surface gradients, say, A_t and X_t.

$$(6.19) \quad \int_{S^*} A_t \times X_t \cdot dS = \sum_{k=1}^{p} [\Gamma_k ; X][\hat{\Gamma}_k ; A] - [\hat{\Gamma}_k ; X][\Gamma_k ; A] .$$

Here $\hat{\Gamma}_k$ is the closed curve on S^* which bounds Σ_k. This formula is a consequence of the observation that an arbitrary surface gradient X can be represented as the difference of two vectors, X' and X'', where curl $X' = 0$ in D and curl $X'' = 0$ in the complement of D (in this way an X_t can be constructed with all its surface periods arbitrary and different from zero). To be sure, (6.19) can be obtained more directly by integrating the (local) identity

$$\text{curl } (\phi \nabla_S \psi - \psi \nabla_S \phi) = 2 \nabla_S \phi \times \nabla_S \psi \quad \text{over the}$$

surface with $\Gamma_k, \hat{\Gamma}_k$, $(k = 1, \ldots, p)$ introduced as cuts.

Formula (6.19) is a two-dimensional (surface) analogue of the previous three-dimensional inner product formulas. The surface gradient, A_t, has its surface curl zero and the combination $X_t \times dS$ where X_t is a surface gradient is easily seen to have zero surface divergence, properly defined. The analogy is somewhat hidden since (6.19) applies to a closed 2-manifold, whereas the three-dimensional formulas referred to open 3-manifolds with boundary. The analogous result would be like formula (9.3) with the summations extending over the full groups, e.g., by considering the open periods as closed by some sort of boundary identification.

7. Two-dimensional Harmonic Fields

Cylindrical Problems[18].

The theory of two dimensional harmonic vectors is particularly elegant. The loss of one dimension leads to a homology theory which is not merely a specialization of the three-dimensional homology but has its own intrinsic symmetries. For example, the homology properties of closed surfaces and closed curves are unrelated in three dimensions but the closed curves in the two-dimensional case are the counterparts of both the closed curves and the (projection of) closed surfaces in the three-dimensional case. This fact is reflected in duality relations which virtually halve the work of formulating the theory of plane harmonic fields and their kin.

The planar theory. Duality in the plane.

As before, our attention is devoted to the solution of problems of the form

$$(7.1) \qquad \begin{cases} \operatorname{div} Z = \sigma \\ \operatorname{curl} Z = \tau \end{cases}$$

where the divergence and curl are now operators in the plane, and the scalar fields, σ, τ are assumed to be given data of the problem[19]. In the plane, however, there exists a duality between the divergence and curl which permits a considerable simplification of the problem. To each plane vector field Z we may associate a dual orthogonal field $\hat{Z}$, where, if Z is given componentwise in a cartesian representation by $Z = (Z_1, Z_2)$ then $\hat{Z}$ is defined by

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[18] This section may be omitted or postponed without disturbing the continuity of exposition.

[19] Since the planar curl has only one component it may be treated as a scalar.

$$\hat{Z}_1 = Z_2 \, , \qquad \hat{Z}_2 = - Z_1 \; . \tag{7.2}$$

With this convention the duality of the vector operators is exhibited by

$$\left\{ \begin{aligned} \text{curl } Z &= \text{div } \hat{Z} \\ \text{div } Z &= - \text{curl } \hat{Z} \; . \end{aligned} \right. \tag{7.3}$$

The divergence theorem in the plane for Z will then prove to be the same as Stokes' theorem for the dual vector $\hat{Z}$.

We shall let Σ denote a plane domain since this notation will be consistent with the notation for open surfaces in three-dimensional problems. For a simply connected plane domain Σ with closed bounding curve Γ, the integral theorems take the form

$$\begin{aligned} \int_{\Sigma} \text{div } Y \, dS &= \oint_{\Gamma} Y \cdot n \, ds \\ \int_{\Sigma} \text{curl } X \, dS &= \oint_{\Gamma} X \cdot t \, ds \end{aligned} \tag{7.4}$$

where $n = \hat{t}$ denotes the exterior normal to Γ. The extension of the integral theorems to more complicated topologies is greatly simplified by the duality relations. To this end let us investigate the homology properties of plane domains.

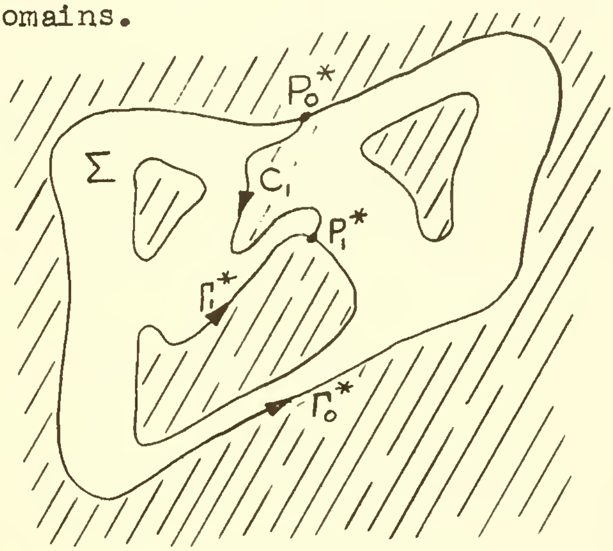

Figure 11. Two-dimensional homology.

Let $\sum$ be a plane domain bounded by a finite number of simple closed curves $\Gamma_0^*, \Gamma_1^*, \ldots, \Gamma_r^*$ oriented so that $\Gamma_0^* + \Gamma_1^* + \ldots + \Gamma_r^* \sim 0$. The closed curves $\Gamma_1^*, \ldots, \Gamma_r^*$ serve as a basis for the closed curve homology group $\{\Gamma\}$ in $\sum$. For a basis of the open arc homology we may choose arcs $C_1, C_2, \ldots, C_r$ where each arc C_k connects a point P_0^* on Γ_0^* to a point P_k^* on Γ_k^*, (Figure 11).

The analogy between the plane homologies and the three-dimensional homologies is easily visualized in a cylindrical domain D which has $\sum$ as its plane cross-section. The closed curves in Σ are the projections of both the bounded closed curves in D and of the closed surfaces which have infinite cylinders as a basis. However, the closed curve group in D has an additional basis element which may be taken as an infinite line perpendicular to Σ. In Σ, the dual group to the closed curves are the open arcs, and these correspond to projections of both the open arcs and the open surfaces which have as a basis infinite cylindrical ribbons. In addition, the open surface group in D has Σ itself as a basis element.

The results relating periods and boundary values for an irrotational X are formally identical with the results in three dimensions (Table 1a, Section 3, page 44). Moreover, it is completely unnecessary to put the results for a solenoidal vector Y (Table 1b) in planar form. It is sufficient to write the table for an irrotational vector in terms of the dual of Y through div Y = - curl $\hat{Y}$, $Y_n = -\hat{Y}_t$. We have only to observe for a given curve with unit tangent t and normal $n = \hat{t}$ that

$$Z_t = \hat{Z}_n , \qquad Z_n = - \hat{Z}_t$$

to obtain relations between the periods involving the boundary normal component of a vector and those involving the tangential normal component of the dual. In particular, the $\mathcal{M}$-periods of Z are given by

$$\oint_{\Gamma_k^*} Z_t \, ds = [\Gamma_k^*; Z]$$

$$\int_{C_k} Z_n \, ds = - \int_{C_k} \hat{Z}_t \, ds = - [C_k; \hat{Z}] \quad ,$$

and the $\mathcal{E}$-periods by

$$\oint_{\Gamma_k^*} Z_n \, ds = - \oint_{\Gamma_k^*} \hat{Z}_t \, ds = - [\Gamma_k^*; \hat{Z}]$$

$$\int_{C_k} Z_t \, ds = [C_k; Z] \quad .$$

The existence theorems also dualize quite nicely. As before, the solution of the system (7.1) is decomposed into solutions of the systems

$$(7.5) \qquad \begin{cases} \operatorname{div} Z = 0 \\ \operatorname{curl} Z = 0 \end{cases}$$

$$(7.6) \qquad \begin{cases} \operatorname{div} Z = \sigma \\ \operatorname{curl} Z = 0 \end{cases}$$

$$(7.7) \qquad \begin{cases} \operatorname{div} Z = 0 \\ \operatorname{curl} Z = \tau \end{cases}$$

For (7.6) we have the special solution

$$(7.8) \qquad \begin{cases} Z = \nabla \phi \\ \phi = \frac{1}{2\pi} \int_{\Sigma} \sigma \log r \; dS \, , \end{cases}$$

and by putting (7.7) into the dual form we have the solution of (7.7) directly from (7.8) in the form

$$(7.9) \qquad \begin{cases} Z = \hat{\nabla} \psi \\ \psi = -\frac{1}{2\pi} \int_{\Sigma} \tau \log r \; dS \, . \end{cases}$$

The functions ϕ and ψ correspond to the concepts of velocity potential and stream function in steady incompressible plane flow. In analogy to (5.7) we see that an arbitrary plane vector may be written in the form

$$(7.10) \qquad Z = H + \nabla \phi + \hat{\nabla} \psi \, ,$$

as the sum of a harmonic vector, a gradient and the dual of a gradient.

By virtue of the representation (7.10) the difficulties in formulating a well-posed problem for (7.1) are relegated to the harmonic part. Let us then consider harmonic vectors in the domain Σ. In the simplest case where Σ is the interior of a simple closed curve Γ^* there is a single-valued potential ϕ with $Z = \nabla \phi$ and $\nabla^2 \phi = 0$. The Laplace equation for ϕ has a unique solution if, for example, either ϕ is given on Γ^*

or $\partial\phi/\partial n$ subject to $\oint_{\Gamma^*} \frac{\partial\phi}{\partial n} ds = 0$. It follows that a unique Z exists either with prescribed boundary values Z_t subject to $\oint_{\Gamma^*} Z_t \, ds = 0$, or boundary values Z_n subject to $\oint_{\Gamma^*} Z_n \, ds = 0$. It is no accident that the compatibility relations for the two problems are similar. The problem for harmonic Z with tangential boundary values is identical with the problem for $\hat{Z}$ with normal boundary values. More generally when the boundary is given by

$$\Gamma^* = \Gamma_0^* + \Gamma_1^* + \dots + \Gamma_r^* \sim 0$$

a harmonic vector will possess two kinds of closed curve periods:

$$\begin{aligned} [\Gamma;Z] &= \oint_{\Gamma} Z_t \, ds \quad \left(= \oint_{\Gamma} \hat{Z}_n \, ds\right) \\ [\Gamma;\hat{Z}] &= \oint_{\Gamma} \hat{Z}_t \, ds = -\oint_{\Gamma} Z_n \, ds \quad . \end{aligned} \tag{7.11}$$

Tangential boundary values Z_t may be freely prescribed on Γ^* subject to the compatibility requirement

$$[\Gamma^*;Z] = \oint_{\Gamma^*} Z_t \, ds = 0 \quad ; \tag{7.12}$$

similarly, the prescription of normal boundary values is subject to the compatibility condition

$$[\Gamma^*;Z] = -\oint_{\Gamma^*} Z_n \, ds = 0 \quad . \tag{7.13}$$

Homogeneous boundary conditions yield open arc periods defined as functions on the open arc homology group, namely,

$$(7.14)\qquad \begin{aligned} [C;Z] &= \int_C Z_t \, ds && (Z_t = 0 \text{ on } \Gamma^*) \\ [C;\hat{Z}] &= -\int_C Z_n \, ds && (Z_n = 0 \text{ on } \Gamma^*) \end{aligned}$$

There is no longer any distinction between the $\mathcal{E}$ group and the $\mathcal{M}$ group; yet a shadow of the distinction remains in the designation of the $\mathcal{M}$-periods of Z by $[\Gamma;Z]$ and $[C;\hat{Z}]$ and the $\mathcal{E}$-periods by $[\Gamma;\hat{Z}]$ and $[C;Z]$, but the $\mathcal{M}$-periods of Z are $\mathcal{E}$-periods of $\hat{Z}$ and vice versa.

Quite as before, if inhomogeneous boundary conditions are given the concept of open period must be extended and the values of the periods on a homology class are defined in terms of the value on a fixed representative and the known boundary values. With this extension of the concept of period we have the plane analogue to Theorem 1:

Theorem 1': There exists a unique harmonic vector Z in Σ with arbitrarily prescribed $\mathcal{M}$-periods (either $[\Gamma;Z]$ or $[C;\hat{Z}]$) and boundary values Z_n on Γ^* subject only to the compatibility condition (7.13). The dual of this theorem is the analogue of Theorem 2, namely:

Theorem $2'$: There exists a unique harmonic Z in Σ with arbitrary $\mathcal{E}$-periods $([\Gamma;\hat{Z}]$ or $[C;Z])$ and boundary values Z_t on Γ^* subject only to (7.12).

Again, by extending the concept of period to the inhomogeneous equations (7.1) so that the values of the periods on a homology class are calculated from the value on a fixed representative, the known functions σ and τ, and the prescribed boundary values we obtain the analogue to Theorem 3:

Theorem $3'$: The system (7.1) possesses a unique solution with arbitrarily prescribed $\mathcal{M}$-periods and normal boundary values Z_n subject to $\oint_{\Gamma^*} Z_n \, ds = \int_{\Sigma} \sigma \, dS$ or with arbitrarily prescribed $\mathcal{E}$-periods and tangential boundary values Z_t subject to $\oint_{\Gamma^*} Z_t \, ds = \int_{\Sigma} \tau \, dS$.

Hodge's theorem for the inner product of two vectors, an irrotational X and a solenoidal Y,

$$(7.15) \qquad \begin{cases} \operatorname{curl} X = 0 \\ \operatorname{div} Y = 0 \end{cases}$$

follows as before with the arcs $C_1, C_2, \dots, C_r$ taken as cross-cuts in Σ. For homogeneous normal boundary conditions we obtain

$$(7.16) \quad (X,Y) = \int_{\Sigma} X \cdot Y \, dS = - \sum_{k=1}^{r} [\Gamma_k^*;X] \, [C_k;\hat{Y}] \quad , \qquad (Y_n = 0 \text{ on } \Gamma^*) \; ,$$

as the analog to (6.5). The analogue to (6.6) is simply the dual to (7.16) and is obtained directly from the observation that $(\hat{X},\hat{Y}) = (X,Y)$ in the form

$$(7.17)\quad (X,Y) = \sum_{k=1}^{r} [C_k;X]\,[\Gamma_k^{*};\hat{Y}]\,, \qquad (X_t = 0 \text{ on } \Gamma^{*}).$$

This duality has a curious consequence for harmonic electric and magnetic fields. In the planar case, the inductance and capacitance matrices are essentially inverse, in contrast to the three-dimensional case in which the two are completely unrelated.

The inductance matrix for harmonic B in Σ under the condition $B_n = 0$ is defined by

$$\Phi_k = \sum_{j=1}^{r} L_{jk}\, I \quad ;$$

where $\mu\, I_k = [\Gamma_k^{*};B]$, $\Phi = [C_k;\hat{B}]$. Similarly, the capacitance matrix for harmonic E with $E_t = 0$ is defined by

$$Q_k = \sum_{j=1}^{r} C_{jk}\, V_j$$

where $Q_k = -\kappa[\Gamma_k^{*};\hat{E}]$ and $V_k = [C_k;E]$. Since the problem with homogeneous tangential conditions on E is equivalent to the problem with homogeneous normal conditions on $\hat{E}$ we make the identifications

$$Q_k = -\kappa[\Gamma_k^{*};\hat{E}] = -\kappa\mu\, I_k$$

$$V_k = [C_k;E] = -\Phi_k$$

from which it follows that

$$(7.18) \qquad (C_{jk}) = c^2 (L_{jk})^{-1}$$

where $c^2 = 1/\kappa\mu$. In the three-dimensional case the $\mathcal{E}$ and $\mathcal{M}$ groups generally have different dimensions so that even the orders of the two matrices differ; in the planar case the inductance and capacitance matrices are practically inverses.

The cylindrical theory.

The term two-dimensional may be considered more generally to apply to cylindrical problems rather than purely planar problems. In other words, a two-dimensional problem is one in which the fields and the attendant conditions are independent of one cartesian coordinate, but vectors are permitted a non-vanishing component in the third direction. Let x and y denote the relevant cartesian coordinates and let z be the ignerable coordinate. In problem (5.1) we set

$$Z = Z' + Z^*, \qquad \Xi = \Xi' + \Xi^*$$

where Z^*, Ξ^* are the components in the direction of the z-axis. The problem is easily reduced to the solution of the problems

$$(7.19) \qquad \begin{cases} \operatorname{div} Z' = \sigma \\ \operatorname{curl} Z' = \Xi^* \end{cases}$$

and

$$(7.20) \qquad \operatorname{grad} Z^* = -\hat{\Xi}' \qquad\qquad (\operatorname{curl} \hat{\Xi}' = 0) .$$

The cylindrical problem may then be decomposed into the plane problem (7.19) and the elementary problem (7.20). The cylindrical homology does differ slightly from the plane homology. If we let Σ represent the plane cross-section of the cylinder it is clear that the cylindrical $\mathcal{E}$-group is isomorphic to the two representations of the plane group. On the other hand the cylindrical $\mathcal{M}$-group has an additional basis element which may be taken as Σ itself in the open surface representation or as a single infinite line perpendicular to Σ in the closed curve representation. Coupled with this additional basis element we have a trivial period condition for problem (7.20) which requires the prescription of either $\int_\Sigma Z^* dS$ or $Z^*(x_0)$ for arbitrary x_0 in Σ. However, for the single-valuedness of Z^* in (7.20), the given function Ξ' must satisfy beyond $\operatorname{curl} \hat{\Xi}' = 0$, the stronger condition

$$(7.21) \qquad \oint_\Gamma \Xi'_n \, ds = 0 .$$

for all closed curves Γ in Σ.

The cylindrical electromagnetic equations.

The cylindrical problems for the pre-Maxwell system (1.1) may easily be decomposed according to the above scheme. For example, the pre-Maxwell equations may be solved first for B and then for E by employing the equations

$$(7.22)\qquad \begin{cases} \operatorname{curl} B' = \mu J^*, & \operatorname{grad} E^* = \dfrac{\partial \hat{B}'}{\partial t} \\ \operatorname{div} B' = 0 & \end{cases}$$

$$\begin{cases} \operatorname{grad} B^* = -\mu \hat{J}' & \operatorname{curl} E' + \dfrac{\partial B^*}{\partial t} = 0 \\ (\oint J_n' \, ds = 0) & \operatorname{div} E' = q/\kappa \end{cases}$$

or first for E and then for B by means of the equations

$$\nabla^2 E^* = \mu \frac{\partial J^*}{\partial t}, \qquad \frac{\partial \hat{B}'}{\partial t} = \operatorname{grad} E^*$$

$$(7.23)\qquad \begin{cases} \operatorname{grad} \operatorname{curl} E' = \mu \dfrac{\partial \hat{J}'}{\partial t}, & \dfrac{\partial B^*}{\partial t} = - \operatorname{curl} E' \\ \operatorname{div} E' = q/\kappa & \end{cases}$$

With this decomposition, the existence theorems developed in Section 10 may be immediately interpreted in the cylindrical case. These methods have immediate application to the cylindrical pinch problem treated in Section 10.

8. Variational Methods[20].

We shall now show how to construct a series of variational problems with unique solutions which are exactly the solutions of the differential problems of Theorems 1-3. This will be a convenient way of looking at these questions in a certain class of fluid magnetic stability problems which we shall treat in a later report in this series. In each of the existence theorems a vector is uniquely determined by differential relations, $\operatorname{div} Z = \sigma$ and $\operatorname{curl} Z = \Xi$, and by given boundary values, Z_t or Z_n, and certain period specifications. By dividing the data which uniquely determine Z into two categories to be labeled "admissible" and "variational" (their precise specification will be given later), we shall characterize Z uniquely as that vector of the class having the same admissible data as Z which minimizes a particular functional. In other words, the vector Z will appear as the unique minimizing vector for a variational problem in which the variational conditions play the role of Euler equations and natural boundary conditions. A simple example of a problem of this type is admissible: curl $X = 0$, X_t given

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[20] Cf. Courant-Hilbert: Methods of Mathematical Physics, Interscience (1953), Chap. IV.

variational: div $Y = 0$, $[S;Y]$ given .

The manner in which the functional appropriate to this problem is obtained, will be described below.

Formulation of the variational problem.

The data for Z are divided by the following table into two natural categories according to their relevance in the period conditions (cf. Table I), regardless of their eventual designation as admissible or variational:

Table V

Division of Boundary and Period Data into Curl and Div Classes

a.	curl Z,	Z_t,	$[\Gamma;Z]$,	$[C;Z]$
b.	div Z ,	Z_n,	$[\Sigma;Z]$,	$[S;Z]$

.

For the present, the table is to be considered merely as a mnemonic device in which tangential boundary values and curvilinear periods are associated with curl Z, while normal boundary values and surface periods are grouped with div Z. Corresponding to this division of data the data in a well-posed problem for Z are assigned to two classes of vectors, the curl class consisting of vectors X which agree with Z in those data which fall in line a of the table, and the div class of vectors Y which agree with Z in that part of the data falling in line b. In the example above, the curl class is taken as the admissible class and the div class as the variational class.

In general, corresponding to every set of data which determines a unique vector according to Theorems 1-3, we shall have two distinct variational problems depending upon which one of the curl class or div class is called admissible and the other variational. The two related problems are called conjugate. Further, there is a precise duality between the categories (a) and (b) of Table V: in all the existence theorems and orthogonality relations a statement remains true if curl is replaced by div, tangential condition by normal, etc.

There remains the problem of assigning a variational function; that is, a functional which is to be minimized on the admissible class. Given an association of the data in one of the existence theorems to classes U of admissible vectors and V of variational vectors as outlined above, we prove:

Theorem 6: Given a class of admissible vectors U which contains at least one element, and given a fixed element V_0 of the variational class, the variational function

$$F[U] = \frac{1}{2}(U - V_0, U - V_0) = \frac{1}{2}\int_D |U - V_0|^2 \, dV \tag{8.2}$$

is minimized on the class of admissible U by $U = Z$.

In Theorem 6, it will often be convenient to replace the functional by

$$F[U] = \frac{1}{2}(U,U) - (U,V_0) \quad . \tag{8.3}$$

The replacement is permissible since the two functionals differ only by the constant $\frac{1}{2}(V_0,V_0)$. For the proof of the theorem we shall require the

Lemma: The vector $U - Z$ is orthogonal to $V - Z$ for every choice of U and V.

The vectors $U-Z$ and $V-Z$ are easily seen to satisfy homogeneous conditions and the orthogonality requirements of Table II, page 71, are satisfied and yield the lemma directly. In the example above, for instance, take $X = U$, $Y = V$. We have

$$\operatorname{curl}(U - Z) = 0 \,, \qquad (U - Z)_t = 0$$
$$\operatorname{div}(V - Z) = 0 \,, \qquad [S;V-Z] = 0$$

so that orthogonality is direct from Table IIa.

We then prove Theorem 6 as follows:

$$\begin{aligned} F[U] - F[Z] &= \frac{1}{2}(U-V, U-V) - \frac{1}{2}(Z-V, Z-V) \\ &= \frac{1}{2}(U-Z, U-Z) - (U-Z, V-Z) \\ &= \frac{1}{2}(U-Z, U-Z) \geq 0 \end{aligned}$$

We conclude that $F[U] \geq F[Z]$ and the equality prevails only when $U = Z$.

It should be remarked that although compatibility conditions do not appear explicitly in this argument they are implicit in the assumption that at least one U and one V exist satisfying their respective conditions. As a matter of fact, the compatibility conditions in Theorems 1-3 are characterized abstractly by the simple requirement that there exist at least one member of each, the curl class and the div class.

Conjugate variational problems. The Lagrange multiplier rule.

Let $\mathbf{X}$ and Y now denote vectors corresponding to the categories (a) and (b) of Table V without regard to which is admissible and which variational. The two functionals for the conjugate problems are given in the form (8.3) as

$$(8.4)\qquad \begin{aligned} F_1[X] &= \tfrac{1}{2}\,(\mathbf{X},\mathbf{X}) - (\mathbf{X},Y_0) \\ F_2[Y] &= \tfrac{1}{2}\,(Y,Y) - (Y,\mathbf{X}_0) \end{aligned}$$

for $\mathbf{X}$ and Y respectively, $\mathbf{X}_0$ and Y_0 being fixed elements of the respective classes. These functionals have a common minimizing vector, Z. Adding the two minima and using the lemma of Theorem 6, we obtain

$$(8.5)\qquad F_1[Z] + F_2[Z] = -(\mathbf{X}_0,Y_0) \quad .$$

Since (X_0, Y_0) is fixed, this formula tells us that an upper estimate of the minimum value in one problem yields a lower estimate for the conjugate problem. Furthermore, if X_0 and Y_0 can be chosen orthogonal (say, by choosing one to be identically zero if it satisfies homogeneous conditions), we have the classical theorem in which one minimum is the negative of its conjugate[21].

As a simple example, let us take a case from the corollary to Theorem 1, the problem with closed curve periods:

$$\text{(8.6)} \qquad \begin{array}{rl} \text{admissible:} & \operatorname{div} U = 0 \ , \quad U_n = 0 \\ \text{variational:} & \operatorname{curl} V = 0, \quad [\Gamma_k ; V] = \nu_k \ . \end{array}$$

From (6.5), we have for any U and V in these classes,

$$\text{(8.7)} \qquad (U,V) = \sum_{k=1}^{p} [\Gamma_k ; V] \, [\Sigma_k ; U]$$

so that the functional (8.3) for the problem takes the form

$$\text{(8.8)} \qquad F[U] = \int_D \frac{1}{2} U^2 \, dV - \sum_{k=1}^{p} \nu_k \int_{\Sigma_k} U \cdot dS \ .$$

The fixed vector V_0 no longer appears in the functional. It is represented by the parameters ν_k which determine the variational class explicitly and it is no longer necessary in this case to choose a specific vector V_0

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[21] Courant-Hilbert (loc. cit.).

which takes the values of the periods to define F[U]. If the open surface representation $\sum$ is used to determine the $\mathcal{M}$ group, the problem then takes the form

$$(8.9)\quad \begin{cases} \text{admissible:} \quad \operatorname{div} U = 0 \ , \quad U_n = 0, \quad [\textstyle\sum_k ; U] = \Phi_k \\ \text{variational:} \quad \operatorname{curl} V = 0 \\ F[U] = \displaystyle\int_D \frac{1}{2} U^2 \, dV \ . \end{cases}$$

Here the variational function is simplified by taking as the chosen element of the variational class, $V_0 = 0$. Again the variational function does not depend on V_0. Problem (8.9) is similar to (8.6) in that all conditions are homogeneous except for a set of $\mathcal{M}$-periods, but in one case these periods have been specified as closed and in the other as open. This, of course, necessitates transferring the period conditions from the curl class to the div class in order to keep the grouping of Table V inviolate. Now, applying the classical procedure of the method of Lagrange multipliers, we alter problem (8.9) and look upon div U = 0, $U_n = 0$, as admissibility conditions while $[\sum_k ; U] = \Phi_k$ is a side condition (this is neither an admissible or a variational condition). Now, using the Lagrange multiplier rule we obtain

$$(8.10)\qquad F[U] = \int_D \frac{1}{2} U^2 \, dV - \sum_{k=1}^{p} \lambda_k \int_{\sum_k} U \cdot dS$$

as the variational function. The two problems (i.e., with open and closed periods specified) have been put in the same form, but the functionals (8.8) and (8.10) have slightly different interpretations. In (8.8) the constants ν_k which appear in $F[U]$ are given data. In the Lagrange multiplier problem, the constants λ_k which appear in the functional (8.10) are not prescribed but must be determined afterwards by the side conditions so as to yield the given values of Φ_k.

The example is directly applicable to a magnetostatic field $U = B$. If the boundaries are perfect conductors, with $B_n = 0$ as the appropriate boundary condition, the period $[\Gamma_k; U]$ represents an m.m.f. or the current j_k in the linked circuit $\hat{\Gamma}_k$ while $[\Sigma_k; U]$ is simply the flux linking the circuit loop $\hat{\Gamma}_k$. If the fluxes are prescribed, $[\Sigma_k] = \Phi_k$, then B minimizes the magnetic energy $\int_D \frac{1}{2\mu} B^2 \, dV$. If the currents are given, $[\Gamma_k] = \mu j_k$, then B minimizes the functional $\int_D \frac{1}{2} B^2 dV - \mu \sum_{k=1}^{p} j_k \Phi_k(B)$. In the one case, admissible vectors B must be chosen consistent with prescribed values Φ_k; in the other, the given values j_k do not affect the admissible class but appear as natural boundary conditions on the minimizing vector. The third possibility (the Lagrange multiplier case) is to consider the fluxes as given, but to ignore this restriction in choosing admissible vectors, use the second variational function, and adjust the λ_k to obtain the given Φ_k.

We have seen in this example how the replacement of an admissibility condition by the dual condition (according to Table V) on the variational side leads to a change in the functional of the problem describable in terms of the Lagrange multiplier rule. This is frequently the case, but sometimes requires a generalization of the Lagrange multiplier rule. For this reason, the formulation of Theorem 6 may be considered to generalize as well as justify the Lagrange multiplier rule. (See the appendix).

Variational formulation of the problem curl curl $A = \Xi$; div $A = \sigma$.

Let us consider one more special problem:

$$(8.11)\begin{cases} \text{admissible:} \quad \operatorname{curl} U = \Xi \\ \text{variational:} \quad \operatorname{div} V = 0, \quad V_n \text{ given}, \quad [\Sigma; V] \text{ given.} \end{cases}$$

Let us prescribe in addition that V_n satisfies $[S_k^*; V] = 0$ so that a regular vector potential with $V = \operatorname{curl} A$ exists. From (6.18) we have

$$(U,V) = (A,\Xi) + \int_{S^*} A_t \times U_t \cdot dS$$

whence

$$(8.12)\qquad F[U] = \int_D \tfrac{1}{2} U^2 \, dV - \int_{S^*} A_t \times U_t \cdot dS$$

where the term (A, Ξ) being constant, is dropped. The information V_n given on S^* now appears in the functional disguised in the form of the multiplier A_t; where we note that A_t is not uniquely defined by the given V_n, but that any representation suffices.

By taking the conjugate problem, but keeping the period conditon on the variational side, we obtain a more interesting situation.

$$(8.13)\qquad \begin{cases} \text{admissible:} & \operatorname{div} U = 0, \quad U_n \text{ given} \\ \text{variational:} & \operatorname{curl} V = \Xi, \quad [\Gamma; V] \text{ given.} \end{cases}$$

Under the prescription on U_n, $[S_k^*; U] = 0$, we have $U = \operatorname{curl} A$. The functional appropriate to this problem, when expressed in terms of A, takes the form

$$(8.14)\qquad F[A] = \int_D \tfrac{1}{2} |\operatorname{curl} A|^2 \, dV - \int_D A_t \cdot \Xi \, dV - \int_{S^*} A_t \times U_t \cdot dS \, .$$

There exists a unique minimizing vector U, but the vector potential A is not unique since an arbitrary gradient may be added. To assure the uniqueness of A, we may proceed in several ways by referring to the uniqueness theorems. Supplementary conditions may be imposed, subject to compatibility requirements, such as

(8.15) $$\operatorname{div} A = \sigma\ ,\ A_t,\ [S;A] \text{ or } [C;A] \text{ given}$$

The problem will take the desired form when the $\mathcal{M}$-periods are given in the form $[\Sigma;U]$ rather than $[\Gamma;V]$, thereby putting the period condition on the admissible side. The boundary values A_t must then be chosen compatible with U_n and with periods $[\hat{\Gamma};A]$ compatible with $[\Sigma;U]$. Instead of requiring this compatibility condition, we may delete all reference to U from the problem and leave the information in A_t whence U_n and $[\Sigma;U]$ may be computed. This yields a problem in which A_t can be arbitrarily specified:

(8.16) $$\begin{cases} \text{admissible:} & \operatorname{div} A = \sigma,\ A_t,\ [S;A] \text{ or } [C;A] \text{ given} \\ \text{variational:} & \operatorname{curl}\operatorname{curl} A = \Xi \end{cases}$$

with the variational function given by (8.14). The surface integral is now a constant of the problem and may be deleted, yielding

(8.17) $$F[A] = \int_D \frac{1}{2}\,|\operatorname{curl} A|^2\, dV - \int_D A \cdot \Xi\, dV\ .$$

In the problem (8.16) - (8.17) we have found a variational formulation of Theorem 4.

We remark that the variational problems (8.11) - (8.12) and (8.16) - (8.17) are conjugate (in a somewhat generalized sense, since one involves U and the other, the vector potential of V) since

$$F_1[U] + F_2[A] = \text{const.}$$

where F_1 is defined by (8.12) and F_2 by (8.16).

9. A General Existence Theorem and Related Considerations[22].

The existence theorems 1-3 may be extended to mixed boundary value problems in which either one of Z_n or Z_t is prescribed on each of the disjoint connected components S_k^* of S^* but not necessarily over all S^*. We shall denote by S_n^* that section of S^* on which normal conditions Z_n are specified, and by S_t^* the remainder of S^*, and there we specify tangential boundary conditions Z_t. In particular, if S^* is connected, either S_n^* or S_t^* is vacuous, and we have one of the previously considered cases.

In the event that neither S_n^* or S_t^* is vacuous it is no longer true generally that the open periods are determined on a homology class by the value on a single member. For example, in Figure 6, page 33, let $S_1^* = S_n^*$, $S_2^* = S_t^*$. If $\sum_1'$ is a member of the class of $\sum_1$ such that the two surfaces can be completed on S^* to a closed bounding surface in D only by the addition of part of S_2^* (i.e. of S_t^*), then it is not possible to specify the integral $[\sum_1';Z]$ in terms of the integral $[\sum_1;Z]$ since Z_n is not specified on S_2^*. Similarly, for open arcs C and C' connecting the two boundary components but initiating at different points on S_1^* the integral $[C';Z]$ cannot be expressed in terms of the integral $[C;Z]$ since X_t is not given on S_n^*. Nonetheless, by a simple device it is possible to recover the property that the periods on a homology class can be obtained from the period on any representative of the class and the known boundary values. The method is to utilize a mixed representation for the abstract group consisting of both open and closed elements and to extend the concept of homology modulo the boundary.

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[22] This section may be omitted or postponed without disturbing the continuity of exposition.

Let us first consider the $\mathcal{M}$ group in its dual representations, $\{\Gamma\}$ the group of closed curves in D, and $\{\Sigma\}$ the group of open surfaces $[\text{mod } S^*]$. Let $\Gamma_1, \dots, \Gamma_p$ denote a basis for $\{\Gamma\}$ which we may assume to consist of simple closed curves on S^* and let $\Sigma_1, \dots, \Sigma_p$ be a reciprocal basis. The most easily comprehended method of forming a mixed basis is to take a subset of the Γ_i, say $\Gamma_1, \dots, \Gamma_k$, and to complete the basis with $\Sigma_{k+1}, \dots, \Sigma_p$. In the given boundary value problem $\Gamma_1, \dots, \Gamma_k$ are chosen as a basis of that subgroup of $\{\Gamma\}$ which is spanned by the closed curves of S_t^*. The abstract subgroup for which $\Gamma_1, \dots, \Gamma_k$ is a closed curve basis, is denoted by $\mathcal{M}_t$. The orthogonal complement to $\mathcal{M}_t$, which we denote by $\overline{\mathcal{M}}_t$, is generated by a basis $\Gamma_{k+1}, \ldots, \Gamma_p$ which lies completely on S_n^* since we have assumed each basis element to be connected[24]. The other representation of $\overline{\mathcal{M}}_t$ in terms of the basis of open surfaces $\Sigma_{k+1}, \ldots, \Sigma_p$ has also a simple interpretation in terms of homology. It is intuitively seen that the $\Sigma_{k+1}, \ldots, \Sigma_p$ may be chosen so as not to intersect S_t^*. The group $\mathcal{M}_t$ may therefore be redefined as the homology group of open surfaces in **D** modulo S_n^*; that is, the group of open surfaces with boundaries on S_n^*. Open surface periods are now defined on $\overline{\mathcal{M}}_t$ exactly as before. Given an element in a homology class of $\overline{\mathcal{M}}_t$, the open surface periods for all other members can now be obtained from the known boundary values on S_n^*. For example, in Figure 7, page 36, let S_n^* consist of S_0^* and S_1^*. The curve Γ_2 may be taken as the basis of $\mathcal{M}_t$, the surface Σ_1 bounded by $\hat{\Gamma}_1$ as the basis for $\overline{\mathcal{M}}_t$.

Similarly, for the $\mathcal{E}$ group we let $\mathcal{E}_n$ consist of the subgroup spanned by the closed surfaces constituting

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[24] Note that the orthogonal subgroup to $\mathcal{M}_t$ may be smaller than the subgroup which is spanned by all closed curves of $\{\Gamma\}$ on S_n^*. For example, in Figure 6 with $S_1^* = S_n^*$, $S_2^* = S_t^*$, $\mathcal{M}_t$ has the basis Γ_2, $\hat{\Gamma}_2$ while the orthogonal complement is the null space.

S_n^*. The orthogonal complement $\overline{\mathcal{E}}_n$ may be assigned a basis consisting of all the components of S_t^* but one. The group $\overline{\mathcal{E}}_n$ may also be thought of in its other representation as the homology group of open curves [mod S_t^*] with a basis consisting of open arcs, each connecting the excluded component to one of the other components of S_t^*.

We are now able to state the proper generalization of Theorems 1-3.

Theorem 7: Let D be a domain with boundary $S^* = S_n^* + S_t^*$ with S_n^* and S_t^* disconnected. There exists a unique solution of (5.1) in D taking boundary values X_n on S_n^* and X_t on S_t^* compatible with σ and Ξ, but otherwise arbitrary, and with arbitrary values for a set of periods on $\overline{\mathcal{M}}_t$ and $\overline{\mathcal{E}}_n$.

Note that the given boundary values explicitly specify $[\Gamma;Z]$ on $\mathcal{M}_t$ and $[S;Z]$ on $\mathcal{E}_n$ so that complete sets of periods on both $\mathcal{M}$ and $\mathcal{E}$ are given by the data.

If neither S_n^* or S_t^* is vacuous then there are no compatibility restrictions on Z_n. The requirement on Z_t is that it be compatible with Ξ_n on S_t^*. Specifically if Γ bounds Σ^* on S^* we have $[\Gamma;Z] = \int_{\Sigma^*} \Xi_n \, dS$. Here, as before in the inhomogeneous case, it is necessary to utilize generalized periods which take into account the prescribed boundary values and the prescribed fields σ and Ξ.

Next, we derive the orthogonality relation for the system (6.2) with the mixed boundary conditions

$$(9.1) \quad \begin{cases} X_t = 0 \quad \text{on} \quad S_t^* = \sum_{k=0}^{\nu} S_k^* \\ Y_n = 0 \quad \text{on} \quad S_n^* = \sum_{k=\nu+1}^{q} S_k^* \end{cases}$$

Observing that ϕ is constant on each component of S_t^* we obtain

$$(9.2) \quad (X,Y) = \sum_{k=0}^{\nu} \phi_k \, [S_k;Y] + \sum_{k=1}^{p} [\Gamma_k;X] \, [\Sigma_k;Y]$$

By writing (9.2) in terms of the homology groups $\overline{\mathcal{E}}_n$ and $\overline{\mathcal{M}}_t$ we obtain an invariant form generalizing Hodge's theorem,

$$(9.3) \quad (X,Y) = \sum_{\overline{\mathcal{E}}_n} [C_k;X] \, [S_k^*;Y] + \sum_{\overline{\mathcal{M}}_t} [\Gamma_k;X] \, [\Sigma_k;Y] ,$$

in which the summations are taken over basis elements of the indicated groups. In other words, the inner product of irrotational X and solenoidal Y over D subject to (9.1) is the bilinear product of the two types of $\overline{\mathcal{M}}_t$ periods plus the bilinear product of the two types of $\overline{\mathcal{E}}_n$ periods.

Vectors X,Y satisfying (6.2) and (9.1) will be orthogonal if

$$(9.4)\quad \begin{cases} [C_k;X] = 0 \quad \text{or} \quad [S^*_k;Y] = 0 \quad \text{on} \quad \overline{\mathcal{E}}_n \\ [\Gamma_k;X] = 0 \quad \text{or} \quad [\Sigma_k;Y] = 0 \quad \text{on} \quad \overline{\mathcal{M}}_t \end{cases}$$

With the aid of these orthogonality conditions it is possible to further refine the decomposition of harmonic vectors given in Table III. The most general such decomposition is the following:

Table VI

H'	H''
$H'_t = 0$ on S^*_t $[\Gamma;H'] = 0$ on $\overline{\mathcal{M}}'_t$ $[C;H'] = 0$ on $\overline{\mathcal{E}}'_n$	$H''_n = 0$ on S^*_n $[\Sigma;H''] = 0$ on $\overline{\mathcal{M}}''_t$ $[S;H''] = 0$ on $\overline{\mathcal{E}}''_n$

A decomposition of this type exists for each separation of S^* into S^*_t and S^*_n and for each further decomposition $\overline{\mathcal{M}}_t = \overline{\mathcal{M}}'_t + \overline{\mathcal{M}}''_t$ and $\overline{\mathcal{E}}_n = \overline{\mathcal{E}}'_n + \overline{\mathcal{E}}''_n$. That H' and H'' are orthogonal follows immediately from (9.4). Given an arbitrary harmonic H, Theorem 7 yields a unique H' satisfying the criteria of Table VI and the conditions $H'_n = H_n$ on S^*_n, $[\Sigma;H'] = [\Sigma;H]$ on $\overline{\mathcal{M}}''_t$ and $[S;H'] = [S;H]$ on $\overline{\mathcal{E}}''_n$. The vector H'' is given by subtractions.

The decomposition given by Table VI can be described as follows. Given certain data which define a unique harmonic vector according to Theorem 7, we split the boundary data and each type of period data into two sets such that, when supplemented by complementary homogeneous data, each set determines a unique harmonic vector, and the two are orthogonal.

To prove Theorem 6 in general requires only slightly more effort than in the special case. Consider, for example, the variational problem in which the data are divided as follows without specific reference as to which class is admissible and which is variational.

$$(9.5) \quad \begin{cases} \text{curl } X = \Xi,\ X_t \text{ given on } S_t^*,\ [\Gamma; X] \text{ given on} \\ \qquad \overline{\mathcal{M}}_t',\ [C; X] \text{ given on } \overline{\mathcal{E}}_n' \\ \text{div } Y = \sigma,\ Y_n \text{ given on } S_n^*,\ [\Sigma; Y] \text{ given on} \\ \qquad \overline{\mathcal{M}}_t'',\ [S; Y] \text{ given on } \overline{\mathcal{E}}_n''. \end{cases}$$

To exhibit the orthogonality of X-Z and Y-Z we obtain from Tables IV and VI

$$(9.6) \qquad Z = \text{curl } A + H' + H'' + \text{grad } \phi$$

and

$$\begin{cases} X = \operatorname{curl} A + \bar{H}' + H'' + \overline{\operatorname{grad} \phi} \\ Y = \overline{\operatorname{curl} A} + H' + \bar{H}'' + \operatorname{grad} \phi \end{cases}$$

where the unbarred symbols represent quantities which are uniquely determined by the data in (9.5) defining the classes X and Y, and the barred symbols depend upon the particular choice of X and Y. For example, expanding Y according to Table IV, the first column alone takes the condition $\operatorname{div} Y = \sigma$ and this with the other conditions in the column determine $\operatorname{grad} \phi$ uniquely. From the decomposition (9.7) it is clear that Z is the only element common to both classes X and Y. The orthogonality $(X-Z, Y-Z) = 0$ is immediate from (9.6) and (9.7). Theorem 6 is thereby proved for the general Theorem 7. The problem with mixed boundary conditions can now be put as a variational problem of the type of Section 8.

10. The Electromagnetic Equations.

We shall now apply the existence theorems of Section 5 to the formulation of well-posed problems for the pre-Maxwell electromagnetic equations. First by employing $\partial B/\partial t$ instead of B as the fundamental variable, the application of the existence theorems is made immediate. Explicit time dependence is next introduced by means of Faraday's law. Most significant for fluid magnetics is the time dependent problem in which only enough data are prescribed to determine B. The results are then applied to a few simple examples, namely Faraday's law for a physical wire, the homopolar generator and the pinch effect.

Well-posed problems for E and $\dot{B}$.

We shall not begin directly with the pre-Maxwell system (1.1) but instead take the simpler system obtained by removing time from the equations, taking $\dot{B} = \partial B/\partial t$ as the fundamental variable instead of B. Differentiating with respect to time in $\operatorname{div} B = 0$ and $\operatorname{curl} B = \mu J$, we obtain the system

$$(10.1)\qquad \begin{cases} \operatorname{curl} E = -\dot{B} & \operatorname{curl} \dot{B} = \mu \dot{J} \\ \operatorname{div} E = q/\kappa & \operatorname{div} \dot{B} = 0 \end{cases}$$

where $\dot{J}$ and q are assumed to be given. The time dependence has been eliminated in the sense that we can solve for $\dot{B}$ and E at any fixed instant, given appropriate boundary conditions and period data, provided that at the given time q and $\dot{J}$ are known as functions of space. For example, boundary conditions on $\dot{B}$ might be obtained by differentiating prescribed boundary data for B. The situation here is similar to that in irrotational incompressible fluid flow where the flow is determined at each instant by the instantaneous boundary values and varies in time only if the boundary conditions change. An interesting special case in which this formulation is identical with the system (1.1) is that for which time may be separated out either sinusoidally or exponentially.

With the observation that $\operatorname{curl} \operatorname{curl} E = -\mu \dot{J}$ we have from Theorem 4:

Theorem 8: There exists a unique solution of (10.1) for E and $\dot{B}$ possessing arbitrary boundary values E_t and $\mathcal{E}$-periods in either representation, closed surface $[S;E]$ (compatible with q), or open curve $[C;E]$.

Since $-E$ is the vector potential of $\dot{B}$, the problem (10.1) is the same as the variational problem of (8.11) et. seq. for $\dot{B}$. In particular, Theorem 8 corresponds to the variational problem (8.16) - (8.17). To recapitulate, the solution for E is given by minimizing

$$F[\mathbf{E}] = \int_D \frac{1}{2} |\mathrm{curl}\ \mathbf{E}|^2\, dV + \int_D \mathbf{E} \cdot \mu \dot{J}\, dV \tag{10.2a}$$

subject to the admissibility condition

(10.2b) $\mathrm{div}\ E = q/\kappa$, E_t given on S^*, $[S;E]$ or $[C;E]$ given.

The value of $\dot{B}$ is then given as $\dot{B} = -\mathrm{curl}\ E$, or it may be found directly as the solution of a problem of the form (8.11) - (8.12) by minimizing

$$F[\dot{B}] = \int_D \frac{1}{2} \dot{B}^2\, dV + \oint_{S^*} E_t \times \dot{B} \cdot dS \tag{10.3a}$$

subject to the admissibility condition

$$\mathrm{curl}\ \dot{B} = \mu \dot{J} . \tag{10.3b}$$

Only the given boundary values E_t appear in the variational function $F[\dot{B}]$ which is therefore well-defined.

A second existence theorem is obtained by applying Theorem 3 to $\dot{B}$ and E in turn.

Theorem 9: A unique solution of (10.1) exists with given boundary values $\dot{B}_n$ and $\mathcal{M}$-periods ($[\sum;\dot{B}]$ or $[\Gamma;\dot{B}]$) as well as given boundary values E_n and $\mathcal{M}$-periods ($[\sum;E]$ or $[\Gamma;E]$).

In physical applications some of the period conditions are seldom encountered. For the magnetic field the $\mathcal{M}$-periods in both versions, the m.m.f.'s ,

$[\Gamma;B]$ and fluxes $[\Sigma;B]$ are generally relevant to applications, while the $\mathcal{E}$-periods either vanish in the form $[S;B]$ or are seldom relevant in the form $[C;B]$ as difference in magnetostatic potential. For the electric field the e.m.f.'s $[\Gamma;E]$ and charges $[S;E]$ are both generally relevant, while the open periods $[C;E]$ as differences in electrostatic potential are sometimes relevant and open surface periods $[\Sigma;E]$ as electric fluxes apparently never are.

Well-posed problems for E and B.

Returning to the electromagnetic equations (1.1) with B as variable rather than $\dot{B}$ we consider the crucial equation

$$\frac{\partial B}{\partial t} + \mathrm{curl}\ E = 0$$

as being the only one in which time variation enters explicitly and also the only one in which E and B are coupled. Let Σ be a surface moving at velocity u and let Γ be its bounding curve. From the identity

$$(10.4)\quad \frac{d}{dt}\int_{\Sigma} Z\cdot dS = \int_{\Sigma}\left(\frac{\partial Z}{\partial t} + u\ \mathrm{div}\ Z\right)\cdot dS - \oint_{\Gamma} u\times Z\cdot dx$$

for the rate of change of the flux of an arbitrary vector field Z through S, setting $Z = B$, we obtain Faraday's law,

$$(10.5)^{25} \qquad \frac{d}{dt}\int_{\Sigma} B\cdot dS = -\oint_{\Gamma} (E + u\times B)\cdot dx \,.$$

We note that previously the analysis held as well for moving boundary as for fixed boundary; any motion of the boundary was irrelevant; the analysis applied wholly to the configuration which obtained at a given instant. With the utilization of (10.5) on the boundary of the domain, however, compatible boundary values of B and E involve the motion of the boundary. Nothing new is obtained from (10.5) when applied to the interior of the domain; it merely expresses a property of any solution of the equations.

For a closed curve Γ on S^* there are two cases to consider: either the surface Σ bounded by Γ may be taken as part of S^* (that is Γ is homologous to zero on S^*) or Σ belongs to one of the homology classes of $\{\Sigma\}$. If $\Gamma \sim 0$ on S^*, (10.5) relates the boundary values $(E + u\times B)_t$ to the values of $\dot{B}_n$. In fact, $\dot{B}_n$ is completely determined by $(E + u\times B)_t$, but if $\dot{B}_n$ is known as a function of time, $(E + u\times B)_t$ is determined only to within an arbitrary additive surface gradient. In the second case with Σ in $\{\Sigma\}$,

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25. An extension of (10.5) to open arcs which is sometimes useful is the following easily derived formula

$$\frac{d}{dt}\int_C (A\cdot dx) = -\int_C (E + u\times B)\cdot dx + \int_C d(u\cdot A).$$

we conclude that boundary values of $(E + u \times B)_t$ are related to the values of the periods $[\Sigma ; B]$, specifically

$$\frac{d}{dt} [\Sigma_k ; B] = - [\hat{\Gamma}_k ; E + u \times B] ,$$

where $\hat{\Gamma}_k$ is the boundary of Σ_k in the notation introduced in Section 3. In other words, if B_n and $[\Sigma ; B]$ are known at $t = 0$ and $(E + u \times B)_t$ is known for $t \geq 0$, then we can compute B_n and $[\Sigma ; B]$ for $t \geq 0$. The converse is not true. Knowledge of B_n for all time only yields $(E + u \times B)_t$ to within an arbitrary surface gradient and $[\Sigma ; B]$ fixes only half of the periods of this surface gradient, namely, $[\hat{\Gamma}]$ but not $[\Gamma]$.

It is sometimes helpful to adopt the interpretation of $(E + u \times B)_t$ as the surface e.m.f. measured in a coordinate system which is instantaneously moving with the local velocity of the surface at the point observed. We recall that this is a Galilean not a Lorentz invariant interpretation (cf. footnote 2, p. 10).

The following immediate modifications of Theorems 8 and 9 apply directly to the pre-Maxwell equations (1.1). The boundary, S^*, of the domain is assumed to have an arbitrary but given velocity u which can vary over the surface and in time.

Theorem 10: A unique solution of the pre-Maxwell system (1.1) for E and B for $t \geq 0$ exists, taking given boundary values E_t or $(E + u \times B)_t$ and periods $[S;E]$ (or $[C;E]$ in case the boundary condition is E_t), all assigned arbitrarily for $t \geq 0$, together with boundary values B_n and $\mathcal{M}$-periods, $[\Gamma;B]$ or $[\Sigma;B]$, assigned only at $t = 0$. The only compatibility condition is $\int_{S^*} B_n \, dS = 0$.

If E_t is given, we refer to Theorem 8 to solve for $\dot{B}$ and E, and then compute B from its value at $t = 0$ which is determined by the given initial data (Theorem 3). If $(E + u \times B)_t$ is given, we first compute from the known initial values the boundary values B_n and periods $[\Sigma;B]$ as functions of time, as discussed in connection with (8.5). Then, B is computed at each instant using these values and Theorem 3. Finally, E_t is computed as $(E + u \times B)_t - (u \times B)_t$ and then E is obtained by Theorem 4. Compatibility is trivially verified at each stage.

Theorem 11: The identical statement is made for (1.1) as in Theorem 9, with $\dot{B}$ replaced by B, and all boundary and period data specified as functions of time, $t \geq 0$.

Note that the data specified here exceeds that specified in Theorem 9 by the inclusion of B_n and its periods at $t = 0$.

Theorem 12: A unique solution of (1.1) exists for the vector B if we specify B_n for $t \geq 0$, $[\Gamma; B]$ or $[\Sigma; B]$ at $t = 0$, and the e.m.f.'s $[\hat{\Gamma}_k; E + u \times B]$, $(k = 1, \ldots, p)$ for all time.

This is a special case of Theorem 10 in which only enough data has been specified to determine B; E will not be unique. In particular, we conclude that insofar as B is concerned, the surface distribution of the "applied voltage" $(E + u \times B)_t$ on the boundary is immaterial; only total e.m.f.'s (and only half of these) are relevant.

An important general remark is that, despite the appearance of a time derivative $\partial B/\partial t$ in the differential equations, one cannot give initial values of B, or of any other quantity as functions of space over the domain D, but one can only give initial values of certain constants (periods) and, on occasion, boundary values.

This theory will be illustrated by a few simple examples. Suppose E_t is a given surface gradient on the surface of a torus. If $B_n = 0$ initially, we conclude that $B_n = 0$ always. The complete electromagnetic field inside (outside) the torus becomes uniquely determined when we specify the initial value of the total magnetic flux inside (outside). Of the two given periods $[\Gamma;E]$ and $[\hat{\Gamma};E]$, one determines the rate of growth of the magnetic field inside the torus, and the other, outside. We may replace the given E_t by any other surface gradient with the same periods without changing B; the electric field E in D will change, of course. In other words, B is insensitive to the precise way in which E is applied.

If the torus is in motion and $(E + u \times B)_t$ is a specified surface gradient, the situation is exactly the same. In particular, if the e.m.f.'s vanish, $[\Gamma_k;E + u \times B] = [\hat{\Gamma}_k;E + u \times B] = 0$, then the two fluxes (inside and outside) are constant in time. We can compute B at each instant knowing only the constant value of the flux and the instantaneous shape of the torus; the velocity of the torus is irrelevant.

This conclusion is, of course, not restricted to a torus. In any topology, if the particularly simple boundary condition $(E + u \times B)_t = 0$ is satisfied, the magnetic field is determined at each instant by the geometrical configuration alone. The electric field will depend also on the instantaneous velocity of the boundary.

Perfect conductors. Faraday's law for a wire.

The property of vanishing e.m.f.'s is attributed to a perfect conductor. In fact, we define a perfect conductor precisely by the constraint

$$E + u \times B = 0 \quad . \tag{10.6}$$

For the present purpose, we assume that the flow velocity $u(x,t)$ of the perfectly conducting fluid is given a priori; in fluid magnetics problems it will be determined together with the electromagnetic field by (1.1), (10.6), and appropriate hydrodynamical equations of the fluid.

Combining (1.1) and (10.6), we have

$$\frac{\partial B}{\partial t} = \text{curl } (u \times B) \quad , \tag{10.7}$$

from which it is possible to conclude that B is determined uniquely for $t > 0$ when its value at $t = 0$ is given. We can now compute E from (10.6) and so derive the values of q as well as J. In other words, a well-posed problem for a perfectly conducting fluid with flow velocity known consists in giving only the initial values of B and requires no information about q or J. In particular, the values of q and J which result may not satisfy conservation of charge (cf. remarks of Section 1).

The integrated form (10.5) of $\partial B/\partial t + \text{curl } E = 0$ states that $(E + u \times B)_t$ is continuous across a discontinuity surface[26]. In particular, it is zero next to a perfect conductor, and this can be taken as an appropriate boundary condition for a domain (e.g., vacuum) bounded by a perfect conductor. From the integral form of $\operatorname{div} B = 0$ we conclude that B_n is continuous across an interface, and, therefore, that the magnetic flux through an element of surface is constant in time at a given point on the perfectly conducting boundary, i.e., at a point moving with the boundary (cf. (10.7)). However, this boundary condition is not independent of the previous one and could be deduced immediately from $(E + u \times B)_t = 0$ and (10.5) as the compatible boundary condition on B_n. In particular, the compatible boundary conditions $B_n = 0$, $(E + u \times B)_t = 0$ are frequently used at the surface of a perfect conductor.

A moving perfect conductor provides a means of actually constructing a physical situation which is appropriate to Theorem 12 in all generality. If we have a perfectly conducting fluid (e.g. a dish of mercury) with $B_n \neq 0$, then, insofar as the motion can be controlled, $E_t = -(u \times B)_t$ can be given quite arbitrary values.

---

[26] We are accepting the integral form as basic rather than the differential form.

Let us now consider an illustration of Faraday's law, (10.5) in the domain exterior to a conducting wire. As stated, (10.5) refers to the flux and e.m.f. of a mathematical closed curve. In practice, the law is frequently used for a physical wire which is almost a closed curve, but is actually cut at one point for the insertion of a voltage-measuring device. For simplicity, assume the wire is a thin (but not one-dimensional) perfect conductor, and the arc joining the ends of the wire, A and B, (Figure 12) is not moving. The wire cannot be made one-dimensional since the differential equations (1.1) do not allow boundary values to be imposed on a one-dimensional manifold. This has the desirable consequence that the perturbation of a given electromagnetic field due to the insertion of a thin

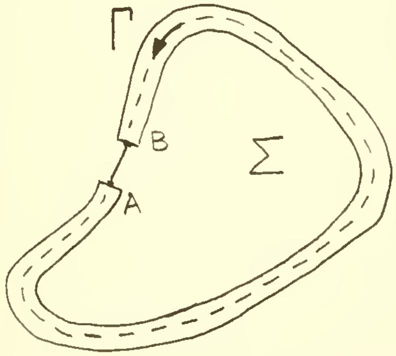

Figure 12. Conducting wire loop.

wire is, roughly speaking, concentrated within a few diameters of the wire. Specifically, $\Phi = \int_{\Sigma} B \cdot dS$ is almost the same for the original and for the perturbed field. Since $(E + u \times B) = 0$ within the perfectly conducting wire, we have

$$\int_A^B E \cdot ds = \frac{d\Phi}{dt} ,$$

where the integral is taken along the short arc AB which closes the circuit. In other words, although the electromagnetic field is perturbed only near the wire, it is very radically changed there, and although $\oint_{\Gamma} (E + u \times B) \cdot dx$ is hardly altered, the distribution along Γ of $(E + u \times B)_t$ is enormously affected by the presence of the wire. We may think of the insertion of the wire as resulting in the transfer of the e.m.f. $\oint_{\Gamma} (E + u \times B) \cdot dx$ to the ends of the wire. Of course, this formula is only approximately valid depending on the finite conductivity, size, shape, and velocity of the wire. The mathematical idealization of a perfectly conducting thin wire is brought home when we realize that closing the loop has the effect of not allowing Φ to change at all!

The homopolar generator.

The ideas above have a simple application in the treatment of the homopolar (or unipolar) generator. Typically, such a generator consists of a perfectly conducting disc rotating about its axis with constant angular velocity ω in the presence of a uniform magnetic field B parallel to the axis of rotation. (The existence of the uniform magnetic field extending across vacuum and conductor is consistent with equation (10.7) for a moving perfect conductor.) An e.m.f. is produced between the axis and the rim of the disc.

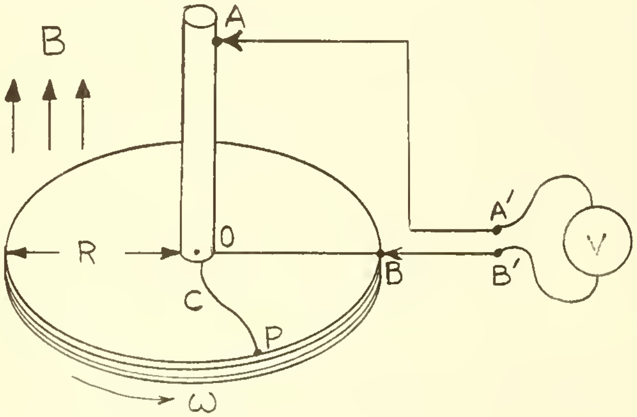

Figure 13. Homopolar generator.

Consider a voltmeter connected by two perfectly conducting leads to a point A touching the axis and a point B touching the edge of the disc; voltmeter and leads are assumed fixed (Figure 13). Let C be a curve which is carried by the disc, and extends from the center to the rim. Since $(E + u \times B)_t = 0$ we have an e.m.f., $\int_C \mathbf{E} \cdot dx = -\int_C (u \times B) \cdot dx$. The magnitude of the e.m.f. from the center to the rim is

$\int_0^R B\,\omega r\,dr = \frac{1}{2}\,\omega R^2 B$. It is asserted that this e.m.f. is exactly what is measured by the voltmeter. To prove the assertion we may use Faraday's law (10.5) either for a moving or fixed circuit. First let us take the path of integration as $A'A\,O\,P\,B\,B'$ where OP designates the moving arc C and PB the circular arc of varying length along the perimeter of the disc. The value of $d\Phi/dt$ linked by the circuit made by this path completed via the voltmeter is B times the rate of change of the covered disc area, $\frac{1}{2}\,\omega R^2$. On the other hand $\int (E + u \times B) \cdot dx$ vanishes on each segment of the open path, so that we obtain across the voltmeter,

$$(10.8) \qquad \int_{A'}^{B'} E \cdot dx = \frac{1}{2}\,\omega R^2 B \ .$$

Alternatively, we may take the fixed path $A'\,A\,O\,B\,B'$ where OB is a fixed segment not moving with the disc. Now we have $d\Phi/dt = 0$. Since $(E + u \times B)_t = 0$ on the surface of the disc, we have $E_t = -(u \times B)_t = \omega r B$ adjacent to the disc and the value of $\int_0^B (E + u \times B) \cdot dx = \int_0^B E \cdot dx$ is $\frac{1}{2}\,\omega R^2 B$. Since the rest of the circuit makes no additional contribution to $\int E \cdot dx$, the result (10.8) follows again.

The electric field E in the entire space is the harmonic vector which takes the given boundary values $E_t = \omega\, r\, B$ on the disc.

The situation is more complex when the magnetic field B is nonuniform. The normal component B_n is constant in time at a given moving point of the boundary (from equation (10.5)). From this boundary condition and whatever other constraints may be imposed on B, we could determine the magnetic field throughout space as a function of time. The e.m.f. $\int_C E \cdot dx = -\int_C (u \times B) \cdot dx$ depends only on the component B_n and hence is constant in time for a curve C carried by the disc. In general, $\int_C E \cdot dx$ depends on the path and not on the endpoints of C alone. Furthermore, since B is a function of time the flux linked by a given circuit and, consequently, the e.m.f. registered at the voltmeter are not easily computed in general. We may, however, take the case in which the wire AA' is very close to the radius OB so that the fluxed linked in the circuit $A'\,A\,O\,B\,B'$ is negligible. Exactly as in the prior instance it follows that the voltmeter registers an e.m.f. of magnitude

$$\int_{A'}^{B'} E \cdot dx = \omega \int r\, B_n(r,t)\, dr$$

where $B_n(r,t)$ gives the values of B_n along the radius OB at the instant t.

Perhaps it should be added that although the magnetic field is "carried" with the conductor in the sense that the value of B_n follows the motion, it is improper to think of individual magnetic lines following the conductor. For suppose we identify a given line by the point at which it intersects the conductor. It will then follow that as the given point on the disc passes under the wire AA', the magnetic line "snaps" from one side of the wire to the other (see Figure 14). In other terms, if a magnetic line intersects two conductors it is clear that it cannot generally be assumed to be carried by both.

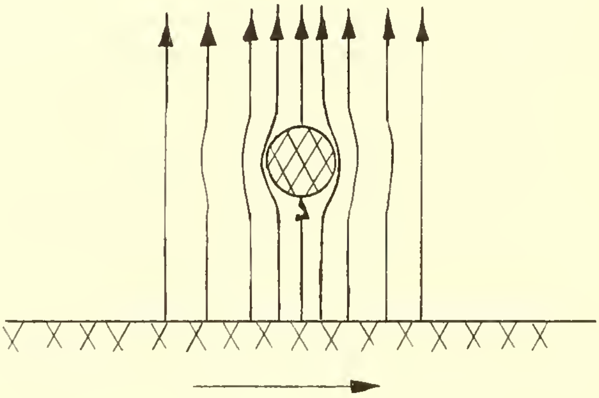

Figure 14. Magnetic lines near conductors.

The argument given above holds exactly for a permanent magnet rotating in its own field as well as for a conducting disc. It is worth remarking also that perfect conductivity is not required for these arguments provided that no currents flow; we would still have the condition $E + u \times B = 0$ in a conductor of finite conductivity. The bar magnet is a case in point; no currents will flow provided that the field of the magnet is axially symmetric.

A very similar situation holds if we have a rotating dielectric instead of a conductor. Let the dielectric constant be denoted by $K = \kappa/\kappa_0$ where κ_0 is the permittivity of free space. Instead of Ohm's law, we have $D = \kappa E_u$ in a frame of observation moving with the dielectric. However, since D does not conventionally include polarization charge, the "applied" field observed in the moving frame is $D = \kappa_0 \, u \times B$. This leads to a polarization charge sufficient to reduce the electric field in the dielectric to $E_u = D/\kappa = u \times B/K$. Returning to the stationary frame (in which the same polarization charge is found, non-relativistically) we observe $E = E_u - u \times B = (\frac{1}{K} - 1)(u \times B)$. For large values of the dielectric constant K, the result is almost the same as for a conductor. Of course, this must be measured by a voltmeter which draws no current.

The cylindrical pinch.

Let us consider the problem of a cylindrically confined plasma (i.e., a perfectly conducting gas) in a cylindrical geometry where the ignorable variable is the axial coordinate z. The plasma cylinder is surrounded by a concentric perfectly conducting shell (Figure 15). Let the variable radius of the plasma be

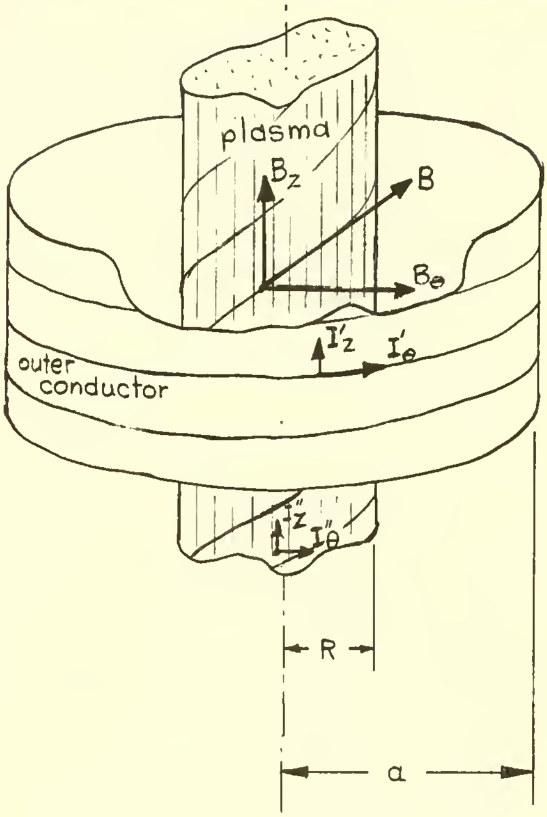

Figure 15. Cylindrically confined plasma.

denoted by R, that of the fixed outer conductor by a. Our problem is to determine R as a function of time under suitably posed conditions. Qualitatively, the situation is described as follows. The vacuum magnetic field B surrounding the plasma is completely determined by the external constraints (e.g., applied e.m.f.'s $[\hat{\Gamma};E]$ maintained by a condenser discharge) and the plasma radius R. The magnetic field provides a driving pressure $B^2/2\mu$ at the surface of the plasma. It is intuitively clear that knowledge of the driving pressure as a function of R and t allows one to compute the motion of the plasma from a given set of fluid equations. A qualitative description can also be given in another way. The forces acting can be computed as above using the magnetic pressure (to which the Maxwell stress tensor is reduced in this example) or by using the formula for the force between current elements. The "pinch" occurs because the current elements in the plasma attract one another. However, one must take care to include the forces due to the currents of the return circuit in the concentric shell and also balance these forces against the fluid pressure of the plasma itself. Quantitatively, the first picture is much easier to apply.

In the vacuum domain D between the two cylinders it is possible to choose as a basis for $\{\Gamma\}$ any line Γ_1 parallel to the axis and any closed curve Γ_2 circling the plasma cylinder. For the open surface group we may form a basis of the surface $\sum_1$, the annular cross-section of D, and $\sum_2$, the infinite plane strip joining the bounding surfaces S' and S'' in an axial section (Figure 16).

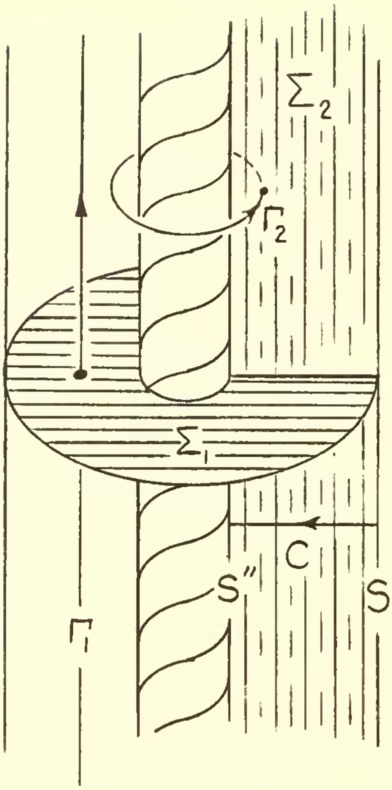

Figure 16. Homology of the domain between two cylinders.

The relevant periods will be fluxes $\Phi_1 = [\Sigma_1; B]$, $\Phi_2 = [\Sigma_2; B]$ or m.m.f.'s $[\Gamma_1; B]$, $[\Gamma_2; B]$. In addition, for a well-posed problem of the type of Theorem 12 it is necessary to prescribe e.m.f.'s $[\hat{\Gamma}_1; E + u \times B]$, $[\hat{\Gamma}_2; E + u \times B]$ where $\hat{\Gamma}_1$ and $\hat{\Gamma}_2$ are the boundaries of Σ_1 and Σ_2, respectively. To avoid infinities in taking integrals on Σ_2 and Γ_1 these will be given for a finite length λ along the axis. Mathematically this corresponds to taking as the domain D the topological torus defined by identifying corresponding points of two transverse sections at a length λ apart.

We shall consider a problem of the type of Theorem 12 in the domain **D.** On the boundary $S^* = S' + S''$ we put $B_n = 0$. A set of $\mathcal{M}$-periods is given initially and the e.m.f.'s $[\hat{\Gamma}_k; E + u \times B]$ are prescribed for all time. Under these conditions, the magnetic field between the plasma and the outer shell is uniquely determined. Physically, this corresponds to the specification of the externally imposed e.m.f.'s, for example, such as might be provided by a condenser discharge and coils. No contribution is made to the periods $[\hat{\Gamma}_k; E + u \times B]$ by integrals on the plasma boundary S'' since $(E + u \times B)_t = 0$ there; the contributions to these periods on the fixed conductor S' will differ from zero only if it is slit and a potential difference applied across the gap.

Proceeding from the existence theorem, the detailed relations between the field and the given periods are easily described. The time derivative of the flux Φ_1 through Σ_1 is given by (10.5) as

$$\dot{\Phi}_1 = \frac{d\Phi_1}{dt} = \frac{d}{dt} \int_{\Sigma_1} B \cdot dS$$

$$= \int_{\Gamma_2''} (E + u \times B) \cdot dx - \int_{\Gamma_2'} (E + u \times B) \cdot dx$$

where the boundary $\hat{\Gamma}_1 = \Gamma_2' - \Gamma_2''$ of Σ_1 is composed of the circles Γ_2' on the outer conductor S' and Γ_2''

on the plasma boundary S''. On S'' we have $(E + u \times B)_t = 0$, and since $u = 0$ on S' we obtain

(10.9) $$\dot{\Phi}_1 = -\oint_{\Gamma_2'} E_t \, ds = -V_\theta'$$

where V_θ' denotes the e.m.f. applied to the circuit Γ_2' around the fixed outer shell and is assumed to be a known function of time. If Φ_1^0 is the initial value of Φ_1 we obtain Φ_1 as a function of time by

(10.10a) $$\Phi_1(t) = \Phi_1^0 - \int_0^t V_\theta' \, (\tau) \, d\tau \, .$$

In the same way, we may calculate the time variation of the flux Φ_2 through a length λ of the ribbon surface $\sum_2$. This will depend on the e.m.f. $V_z'(t) = \oint_{\Gamma_1'} E_t \, ds$ applied to a length λ of the conducting shell, namely,

(10.10b) $$\Phi_2(t) = \Phi_2^0 - \int_0^t V_z' \, (\tau) \, d\tau \quad ,$$

where Φ_2^0 is the initial value of Φ_2.

In this simple geometry the harmonic vector field B can be computed explicitly in terms of either fluxes or currents. The axial component B_z is constant in space and is given by

(10.11a) $$B_z = \frac{\Phi_1}{\pi(a^2-R^2)}$$

Denoting the azimuthal current about a length λ of the outer conductor by I_θ' we have also

$$(10.11b) \qquad B_z = \frac{\mu I_\theta'}{\lambda} ,$$

whence

$$(10.12) \qquad \Phi_1 = \pi(a^2-R^2)B_z = \frac{\pi\mu}{\lambda}(a^2-R^2)\, I_\theta' .$$

The component $B - B_z = B_\theta$ of the magnetic field is harmonic in the plane annular surface Σ_1 and is defined as a function of radius r in terms of the flux Φ_2 by

$$(10.13a) \qquad B_\theta = \frac{\Phi_2}{\lambda r \log\left(\frac{a}{R}\right)} ,$$

or, in terms of the axial current I_z'' in the plasma, by

$$(10.13b) \qquad B_\theta = \frac{\mu I_z''}{2\pi r} .$$

Combining these results we obtain the relation

$$(10.14) \qquad \Phi_2 = \frac{\mu\lambda}{2\pi}\left(\log \frac{a}{R}\right) I_z'' .$$

The inductance matrix obtained from (10.12), (10.14) is diagonal:

$$(\Phi_1, \Phi_2) = \mu \begin{pmatrix} \frac{\pi}{\lambda}(a^2-R^2) & 0 \\ 0 & \frac{\lambda}{2\pi}\log\frac{a}{R} \end{pmatrix} \begin{pmatrix} I_\theta' \\ I_z'' \end{pmatrix} .$$

From the relations (10.9), (10.10) we obtain

(10.15) $V_z' = \frac{d}{dt}[\frac{\mu\lambda}{2\pi}(\log \frac{a}{R})I_z''], \quad V_\theta' = \frac{d}{dt}[\frac{\pi\mu}{\lambda}(a^2-R^2)I_\theta']$.

If, for example, both the current I_z'' and e.m.f. V_z' are experimentally measured, the radius of the plasma R can be determined explicitly as the solution of this first order differential equation. By employing one additional relation which we shall get from the fluid dynamics, we shall be able to express everything in terms of a single type of quantity, e.g., the applied e.m.f.'s as known functions of time.

Thus far we have used no fact about the plasma except perfect conductivity. In a subsequent note we shall show that the quantity $\hat{p} = p + \frac{B^2}{2\mu}$, where p is the fluid pressure, is continuous across the interface S''. In particular, we have

(10.16) $$\hat{p} = p + \frac{(\hat{B}'')^2}{2\mu} = \frac{(B'')^2}{2\mu}$$

where $\hat{B}''$ is the magnetic field at the plasma side of the boundary and B'' is the value at the boundary of the already described field in D. Specifically, from (10.11) and (10.13),

(10.17) $$\begin{aligned}\hat{p} &= p + \frac{1}{2\mu}[(\hat{B}_z'')^2 + (\hat{B}_\theta'')^2] \\ &= \frac{\Phi_1 I_\theta'}{2\pi\lambda(a^2-R^2)} + \frac{\Phi_2 I_z''}{4\pi\lambda R^2[\log a - \log R]}\end{aligned}$$

If we can compute $\hat{p}$ in terms of R and t from the fluid side, we have the desired relation enabling us to solve for all quantities in terms of the applied e.m.f.'s, assumed given, or any other set of given data (e.g., currents).

Since we do not wish here to consider fluid magnetic problems in detail, we proceed by making simplifying assumptions which allow us to obtain $\hat{p}$ as a function of R. Whatever the nature of the fluid, it is clear in general that the complete solution to the problem will be determined by (10.17) if $\hat{p}$ is known as a function of R and t. Let us assume that the motion is so slow that the fluid density ρ is uniform across the plasma. Conservation of mass yields

$$\rho = \rho_0 R_0^2/R^2$$

where ρ_0, R_0 denote initial values for the plasma. If it is further assumed that the adiabatic law holds

$$p = p_0(\rho/\rho_0)^\gamma$$

then we have a relation between p and R:

$$\hat{p} = p_0 \left(\frac{R_0}{R}\right)^{2\gamma} \tag{10.18}$$

We have only to determine the plasma boundary field $\hat{B}''$ to have a relation defining R in terms of the period data in the plasma and vacuum domains.

For simplicity we suppose that the plasma field B is uniform[27]; that is, that the plasma currents lie on the surface S''. Let Φ_3 be the constant flux through the plasma cross-section; hence,

$$\hat{B} = \hat{B}_z = \frac{\Phi_3}{\pi R^2} . \tag{10.19}$$

Putting the given information into (10.16) we obtain

$$\hat{p} = p_0 \left(\frac{R_0}{R}\right)^{2\gamma} + \frac{\Phi_3^2}{2\pi^2 \mu R^4} . \tag{10.20}$$

In a high temperature plasma $\gamma = 2$,[28] and we obtain especially simply

$$p = \hat{p}_0 \left(\frac{R_0}{R}\right)^4 \tag{10.20a}$$

where $\hat{p}_0$ is the initial value of $\hat{p}$. Putting the value of $\hat{p}$ into (10.17) we obtain a relation which defines R ultimately in terms of the e.m.f.'s in D. If B_z vanishes in the domain D between the plasma and the outer conductor the value of R is related directly to the primary plasma current, namely,

$$p_0 \left(\frac{R_0}{R}\right)^{2\gamma} + \frac{\Phi_3^2}{2\pi^2 \mu R^4} = \frac{\mu (I_z'')^2}{8\pi^2 R^2}$$

---

[27] Since the density is assumed uniform, this will be true if B is uniform at any given time.

[28] The gas compresses two-dimensionally if the mean collision time is large.

In case $\gamma = 2$, this equation yields

$$R = 2\pi R_0^2 \sqrt{2\hat{p}_0/\mu}/I_z'' \ .$$

In any event, the relation between the plasma radius R and the currents is an algebraic one.

11. Relation between the Maxwell and pre-Maxwell Systems.

It has already been remarked that the type of data which is appropriate to a well-posed problem for the Maxwell equations is entirely different from that required for the pre-Maxwell system (cf. Section 1). Consequently, it is not immediately evident under what circumstances a solution of the pre-Maxwell system can approximate a solution of the Maxwell system. We shall show in a large class of electromagnetic problems which are well-posed for the Maxwell system, how related well-posed problems for the pre-Maxwell system may be obtained by omitting certain of the specified data which single out a Maxwell solution. In the sense in which the related pre-Maxwell solution is a satisfactory approximation to the Maxwell solution, the set of data which was omitted from the Maxwell problem must be practically irrelevant.

For convenience we denote the solution of a Maxwell problem by E_M, B_M, the solution of the related pre-Maxwell problem by E_P, B_P. The primary question is, in what sense are $E' = E_M - E_P$ and $B' = B_M - B_P$ small. For simplicity we take a fixed bounded domain D and assume $q = 0$, $J = 0$.

One set of prescribed data for a well-posed Maxwell problem consists of initial values of E and B subject to $\operatorname{div} E = 0$ and $\operatorname{div} B = 0$, and tangential boundary values E_t prescribed as a function of time. The related pre-Maxwell problem is defined as follows. The same boundary values E_t are prescribed for E_P as for E_M. It is easily seen that the periods $[S;E_M]$ are constant in time (this is essentially charge conservation) and the values of these periods are computed from the initial values of E_M and assigned to E_P for all time. From the initial values $B(x)$ we take only the boundary values B_n and the periods $[\sum;B]$ (which are well-defined for B_M as well as for B_P since $\operatorname{div} B = 0$ everywhere). This completes a well-posed pre-Maxwell problem according to Theorem 10.

Another well-posed Maxwell problem requires the same initial data as above, together with normal boundary values E_n and B_n (subject to $[S_k^*;B] = 0$, $[S^*;E] = 0$) and periods $[\Gamma;E]$, $[\hat{\Gamma};E]$ all given as functions of time. The closed curve periods are defined in the extended sense (cf. Section 4) and must be given on selected curves $\Gamma_i, \hat{\Gamma}_i$. The given boundary values B_n determine E_t to within an arbitrary local surface gradient and the periods $[\Gamma;E]$, $[\hat{\Gamma};E]$ fix the periods of this surface gradient. The related pre-Maxwell problem is defined by prescribing the same boundary values and periods $[\Gamma;E]$ and $[\hat{\Gamma};E]$ as in the Maxwell problem, together with initial values of the periods $[\sum;B]$ computed from the initial values of B_M (cf. Theorems 9, 11).

We now state the following conjecture. If we let the Maxwell problem M be fixed except that the speed of light, c, is allowed to become infinite, the sequence of solutions $\mathbf{E}_M(c)$, $B_M(c)$ approaches weakly the limit $\mathbf{E}_P$, B_P of the associated problem; (c does not appear in the pre-Maxwell equations). More precisely, the statement is

$$\int_{t_1}^{t_2} \mathbf{E}' \, dt \longrightarrow 0 \, , \qquad \int_{t_1}^{t_2} B' \, dt \longrightarrow 0$$

at any fixed point x and for any fixed time interval (t_1, t_2).

Qualitatively, the above conjecture is that a solution of Maxwell's equations differs from an appropriately chosen solution of the pre-Maxwell equations by a rapidly oscillating function with mean zero but not with zero mean energy. In other words, the solution of Maxwell's equation separates unambiguously into a "slowly varying part" and into electromagnetic waves which propagate with the speed of light. In this context, it is illuminating to note that the solution B_P is at each instant the projection of B_M onto the surbspace of harmonic vectors (it is easily seen that the decomposition $B_M = B' + B_P$ is the representation of B_M as the sum of projections into the classes of Table IV where B and B' correspond to the second and third columns of the table respectively).

A proof of the conjecture could probably be obtained in the special case where the pre-Maxwell solution is time independent, namely for B_n independent of time (or $E_t = 0$). By separating time as a factor $e^{i\omega ct}$ we obtain a solution of the Maxwell problem as an eigenfunction expansion. The first term, $\omega = 0$, is exactly the pre-Maxwell solution; the other terms oscillate more and more rapidly as $c \longrightarrow \infty$ and in the limit will consequently yield mean value zero over any finite time interval.

For an exterior domain, the conjecture can be strengthened; the convergence will be pointwise rather than weak since the initial electromagnetic wave energy will radiate out to infinity.

If the boundary of the domain D is permitted to move, it is likely that the electromagnetic wave component E', B' of E_M, B_M will vary "adiabatically" with the motion of the walls; i.e. the variation in amplitude of each mode can be computed from the work done by the moving wall against the radiation pressure of the given wave component.

Appendix. Note on generalization of the Lagrange multiplier rule.

In Section 8, we have seen how the transfer of an admissibility condition to the variational side (with replacement by the dual condition according to Table V) led to a change in the form of the variational function which may be described by the Lagrange multiplier rule. In general, the change is not so describable and will require a generalization of the rule.

The simplest cases occur when the differential equations and boundary conditions are homogeneous, curl $X = 0$, div $Y = 0$ and either $X_t = 0$ or $Y_n = 0$ on S^*. In these problems, (X,Y) may be written in terms of periods alone, (6.5), (6.6). Take the admissible category as $U = Y$ and the variational category as $V = X$. If periods of X are to be specified, the term (Y,X_0) in the variational function has the exact form of the Lagrange multiplier rule. If periods of Y are specified, X_0 can be taken identically zero and the term (Y,X_0) dropped. The situation is entirely analogous if we take X to be admissible and Y variational. In the more general case of Section 9, we can still express (X,Y) in terms of periods, (9.3), if the differential relations and boundary conditions are homogeneous. Those terms in this bilinear sum in which periods of the variational class are given take the form of the Lagrange multiplier rule, while the terms which involve prescribed periods of the admissible class are constant and may be dropped from the variational function.

In the inhomogeneous cases, the more complicated representations of the inner products (6.15), (6.18) indicate the necessity for introducing appropriate scalar or vector potentials in order to express (U,V) explicitly in terms of the data which define the variational class. If both σ and Ξ differ from zero, a form only vaguely reminiscent of the Lagrange multiplier rule can be obtained. The functional may be expressed in terms of certain "partial" potentials ϕ and A which refer not to U and V, which have no potentials in general, but to components of their orthogonal decompositions according to Table IV, page 73.

If only one of σ or Ξ is zero, two different situations may arise. When the inhomogeneous differential relation is on the variational side, then $F[U]$ has an explicit expression as a functional of a potential of U (usually not single-valued), either as $F[\phi]$ or $F[A]$, rather than of U itself, as in the variational problem of (8.16), (8.17), for example. If, on the other hand, the inhomogeneous relation is in the admissible class, then U itself is the argument of F, but the given inhomogeneous boundary or period data in the variational class appear in the functional as multipliers which are defined in terms of the potentials of a properly chosen variational vector V, rather than in terms of V itself.

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